Optické vlastnosti – fenomenologická teória

A) Nelokálne funkcie odozvy

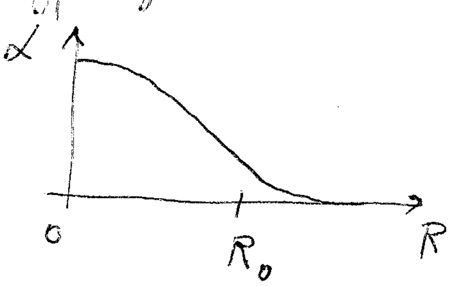
$y(\vec{r},t)$ = odozva v mieste $\vec{r}$, čase t

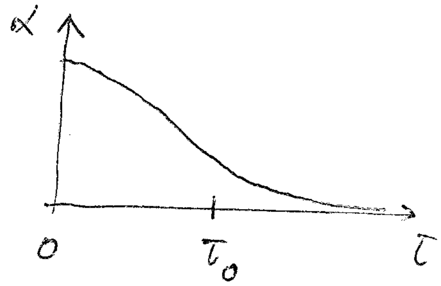
$x(\vec{r},t)$ = príčina —"—

$$\boxed{y(\vec{r},t) = \int d^3R \int_0^{\infty} d\tau\; \alpha(R,\tau)\, x(\vec{r}-\vec{R}, t-\tau)}$$

↓ nelokálna funkcia odozvy

Typicky:





R_0 = typická vzdialenosť, ktorá ovplyvňuje odozvu

τ_0 = typický čas, —"—

Fourier:

$$y(\vec{r},t) = \int \frac{d^3q}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, y_{q\omega}\, e^{i\vec{q}\cdot\vec{r} - i\omega t}$$

$$y_{q\omega} = \int d^3r \int_{-\infty}^{\infty} dt\; y(\vec{r},t)\, e^{-i\vec{q}\cdot\vec{r} + i\omega t}$$

Podobne pre $x(\vec{r},t)$.

$$\rightarrow\; y_{q\omega} = \int d^3r \int_{-\infty}^{\infty} dt\; e^{-i\vec{q}\cdot\vec{r} + i\omega t} \int d^3R \int_0^{\infty} d\tau\; \alpha(R,\tau)\, x(\vec{r}-\vec{R}, t-\tau)$$

$$y_{q\omega} = \int d^3R \int_0^{\infty} d\tau\; \alpha(R,\tau)\, e^{-i\vec{q}\cdot\vec{R} + i\omega\tau}$$

$$\cdot \int d^3r \int_{-\infty}^{\infty} dt\; e^{-i\vec{q}\cdot(\vec{r}-\vec{R}) + i\omega(t-\omega)}\; x(\vec{r}-\vec{R}, t-\tau)$$

Teda $\boxed{y_{q\omega} = \alpha_{q\omega} X_{q\omega}}$ kde $\boxed{\alpha_{q\omega} = \int d^3R \int_0^\infty d\tau\, \alpha(R,\tau)\, e^{-iq\dots}}$

pre Fourierove komponenty je odozva lokálna!

očakávame: $\alpha_{q\omega} \approx \text{const}$ pre $q < \frac{1}{R_0}$ a $\omega < \frac{1}{\tau_0}$

B → $\boxed{\text{Kramers - Kronigove vzťahy}}$

zanedbajme závislosť od $\vec{q}$: $y(t) = \int_0^\infty d\tau\, \alpha(\tau)\, x(t-\tau)$

$$\boxed{\alpha_\omega = \int_0^\infty d\tau\, \alpha(\tau)\, e^{i\omega\tau}}$$

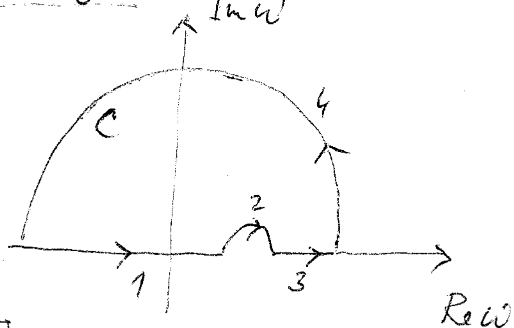
- α_ω je analytická pre $\operatorname{Im}\omega > 0$

  (pre $\alpha(\tau) \to 0$, ak $\tau \to 0$)

- $\alpha_\omega = \alpha'_\omega + i\alpha''_\omega$ (reálna a imaginárna časť)

Potom platí: $\boxed{\begin{aligned} \alpha'_\omega &= \alpha'_{-\omega} \\ \alpha''_\omega &= -\alpha''_{-\omega} \end{aligned}}$

- Počítajme integrál pozdĺž C



$$\boxed{\oint_C \frac{ds\, \alpha(s)}{s-\omega} = 0}$$ (Cauchy)

Predpoklad: $\alpha(\omega) \to 0$ pre $\omega \to \infty$

Potom $\int_4 \frac{ds\, \alpha(s)}{s-\omega} \to 0$, na druhej strane $\int_1 + \int_3 = P\int_{-\infty}^{\infty} \frac{\alpha(s)\, ds}{s-\omega}$

$$\int_2 \frac{\alpha(s)\,ds}{s-\omega} = \int_2 \frac{\alpha(\omega+z)\,dz}{z} \xrightarrow{z=\rho e^{i\varphi}} \alpha(\omega)\int_\pi^0 \frac{i\rho e^{i\varphi}d\varphi}{\rho e^{i\varphi}} = -i\pi\alpha(\omega)$$

Preto máme
$$\boxed{\alpha(\omega) = \frac{1}{\pi i} P\int_{-\infty}^{\infty} \frac{\alpha(s)\,ds}{s-\omega}}$$

$$\boxed{\begin{aligned}\alpha'(\omega) &= \frac{1}{\pi} P\int_{-\infty}^{\infty}\frac{\alpha''(s)\,ds}{s-\omega}\\ \alpha''(\omega) &= -\frac{1}{\pi} P\int_{-\infty}^{\infty}\frac{\alpha'(s)\,ds}{s-\omega}\end{aligned}} \xrightarrow[]{\substack{\alpha'(\omega)=\alpha'(-\omega)\\ \alpha''(\omega)=-\alpha''(-\omega)}} \boxed{\begin{aligned}\alpha'(\omega) &= \frac{2}{\pi} P\int_0^{\infty}\frac{s\,\alpha''(s)\,ds}{s^2-\omega^2}\\ \alpha''(\omega) &= -\frac{2\omega}{\pi} P\int_0^{\infty}\frac{\alpha'(s)\,ds}{s^2-\omega^2}\end{aligned}}$$

Kramers – Kronig

per partes

$$\boxed{\alpha''(\omega) = -\frac{1}{\pi} P\int_0^{\infty} \ln\left|\frac{s+\omega}{s-\omega}\right| \frac{d\alpha'(s)}{ds}\,ds} \quad (*)$$

(c) Elektrodynamika izotropného prostredia – materiálové vzťahy

$E, B \propto e^{i\vec{q}\cdot\vec{r}-i\omega t}$

úloha: nájsť $j_\mu = (\rho, \vec{j})$ pre dané $A_\mu = (\varphi, \vec{A})$

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \nabla\varphi = i(\omega\vec{A} - \vec{q}\varphi) \qquad \nabla \leftrightarrow i\vec{q}$$

$$\vec{B} = \nabla\times\vec{A} = i\vec{q}\times\vec{A} \qquad \frac{\partial}{\partial t} \leftrightarrow -i\omega$$

lineárna odozva: $\boxed{j_\mu = -K_{\mu\nu}A_\nu}$

$$\boxed{\begin{aligned}\rho &= -K_{00}\varphi - K_{0i}A_i\\ j_i &= -K_{i0}\varphi - K_{ij}A_j\end{aligned}} \quad (1)$$

kalibrač. transformácia: $\begin{aligned}\varphi &\to \varphi + \frac{\partial\chi}{\partial t}\\ \vec{A} &\to \vec{A} - \nabla\chi\end{aligned} \longleftrightarrow \boxed{\begin{aligned}\varphi &\to \varphi - i\omega\chi\\ \vec{A} &\to \vec{A} - i\vec{q}\chi\end{aligned}}$

j^μ nezávislé od voľby kalibrácie $\longrightarrow$ $$\boxed{\begin{aligned} K_{00}\,\omega + K_{0i}\,q_i &= 0 \\ K_{i0}\,\omega + K_{ij}\,q_j &= 0 \end{aligned}} \quad (2)$$

rovnica kontinuity

$$\frac{\partial \rho}{\partial t} + \nabla\cdot\vec{j} = 0 \;\longrightarrow\; \omega\rho = \vec{q}\cdot\vec{j}$$

(má platiť $\forall \varphi, \vec{A}$) $\longrightarrow$ $$\boxed{\begin{aligned} \omega K_{00} - q_i K_{i0} &= 0 \\ \omega K_{0i} - q_j K_{ji} &= 0 \end{aligned}} \quad (3)$$

izotropia $\longrightarrow$ $$\boxed{\begin{aligned} &K_{0i},\, K_{i0} \propto q_i \\ &K_{ij} = K\left(\delta_{ij} - \frac{q_i q_j}{q^2}\right) + K'\delta_{ij} \end{aligned}} \quad (4)$$

z rovníc 2, 3, 4 vyplýva:

$$\boxed{\begin{aligned} K_{0i} &= -\frac{\omega q_i}{q^2} K_{00} \\ K_{i0} &= \frac{\omega q_i}{q^2} K_{00} \\ K' &= -\frac{\omega^2}{q^2} K_{00} \end{aligned}}$$

$\longrightarrow$ namiesto 16 funkcií odozvy $K_{\mu\nu}$ iba 2 funkcie: $K_{00}(q^2,\omega)$, $K(q^2,\omega)$

Parametrizácia: $$\boxed{\begin{aligned} K_{00} &= \varepsilon_0 \chi q^2 \\ K &= -\frac{1}{\mu_0}\chi_m q^2 \end{aligned}}$$

Potom rovnica (1) má tvar:

$$\boxed{\begin{aligned} \rho &= -\varepsilon_0 \chi\, i\vec{q}\cdot\vec{E} \\ \vec{j} &= -\varepsilon_0 \chi\, i\omega \vec{E} + \varepsilon_0 \chi_m c^2\, i\vec{q}\times\vec{B} \end{aligned}} \quad (5)$$

$\longrightarrow$ explicitne kalibračne invariantné!

funkcie odozvy $\chi(q^2,\omega)$, $\chi_m(q^2,\omega)$

Pre konštantné χ, χ_m sa (5) redukujú na $$\boxed{\begin{aligned} \rho &= -\nabla\cdot\vec{P} \\ \vec{j} &= \frac{\partial \vec{P}}{\partial t} + \nabla\times\vec{M} \end{aligned}}$$

kde $\boxed{\vec{P} = \varepsilon_0 \chi \vec{E}}$, $\boxed{\vec{M} = \frac{1}{\mu_0}\chi_m \vec{B}}$

Alternatívna formulácia (5):

$$\nabla\times\vec{E}+\frac{\partial\vec{B}}{\partial t}=0 \longrightarrow \vec{B}=\frac{1}{\omega}\vec{q}\times\vec{E} \longrightarrow \text{dosadíme do (5):}$$

$$\boxed{\begin{aligned}\rho &= -\varepsilon_0\chi\, i\vec{q}\cdot\vec{E}\\ j_i &= -i\varepsilon_0\omega\left[\chi\,\delta_{ij}+\frac{\chi_m q^2c^2}{\omega^2}\left(\delta_{ij}-\frac{q_iq_j}{q^2}\right)\right]E_j\end{aligned}} \quad (6)$$

možno písať:

$$\boxed{\begin{aligned}\vec{j}_\parallel &= \sigma_\parallel(q,\omega)\vec{E}_\parallel\\ \vec{j}_\perp &= \sigma_\perp(q,\omega)\vec{E}_\perp\end{aligned}} \quad (7) \qquad \boxed{\begin{aligned}\sigma_\parallel(\vec{q},\omega) &= -i\varepsilon_0\omega\chi\\ \sigma_\perp(\vec{q},\omega) &= -i\varepsilon_0\omega\left(\chi+\frac{q^2c^2}{\omega^2}\chi_m\right)\end{aligned}}$$

v limite $q\to 0$ musí byť

$$\boxed{\sigma_\parallel(0,\omega)=\sigma_\perp(0,\omega)}$$

(D) Vlny v lineárnom prostredí

Maxwell:

$$\boxed{\begin{aligned}&\nabla\times\vec{B}-\frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}=\mu_0\vec{j}_{tot} && \nabla\cdot\vec{E}=\frac{1}{\varepsilon_0}\rho_{tot}\\ &\nabla\times\vec{E}+\frac{\partial\vec{B}}{\partial t}=0 && \nabla\cdot\vec{B}=0\end{aligned}}$$

kde

$$\left.\begin{aligned}\rho_{tot} &= \rho_{ext}+\rho\\ \vec{j}_{tot} &= \vec{j}_{ext}+\vec{j}\end{aligned}\right\} \quad \rho,\vec{j} \text{ dané materiálovými vzťahmi (6, 7)}$$

• Pozdĺžne polia:

$$i\vec{q}\cdot\vec{E}=\frac{1}{\varepsilon_0}(\rho_{ext}+\rho) \longrightarrow \boxed{\varepsilon_0\varepsilon_\parallel(\vec{q},\omega)\, i\vec{q}\cdot\vec{E}_\parallel=\rho_{ext}} \quad (8)$$

$$\boxed{\varepsilon_\parallel(\vec{q},\omega)=1+\chi=1+\frac{i\sigma_\parallel}{\varepsilon_0\omega}} \quad (9)$$

Rovnica (8) popisuje všetky longitudinálne polia.

Spontánne kmity: ak $\boxed{\varepsilon_\parallel(\vec{q},\omega)=0}$
(pozdĺžne)

Priečne polia:

$$\nabla\times(\nabla\times E) = -\frac{\partial}{\partial t}(\nabla\times B)$$

$$\downarrow$$

$$\frac{1}{c^2}\frac{\partial \bar{E}}{\partial t} + \mu_0 \bar{j}_{ext} + \mu_0 \bar{j}$$

$$\bar{\nabla}(\nabla\cdot\bar{E}) - \Delta\bar{E} = -\frac{1}{c^2}\frac{\partial^2\bar{E}}{\partial t^2} - \mu_0\frac{\partial \bar{j}_{ext}}{\partial t} - \mu_0\frac{\partial\bar{j}}{\partial t}$$

zoberme priečnu zložku:

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta\right)\vec{E}_\perp + \mu_0\frac{\partial \bar{j}_\perp}{\partial t} = -\mu_0\frac{\partial \bar{j}_{ext}}{\partial t}$$

Fourier:

$$(q^2 - \omega^2/c^2)\vec{E}_\perp - i\omega\mu_0\sigma_\perp\vec{E}_\perp = i\omega\mu_0\vec{j}_{ext}^{\,\perp}$$

$$\boxed{\left[q^2 - \varepsilon_\perp(q,\omega)\frac{\omega^2}{c^2}\right]\vec{E}_\perp = i\omega\mu_0\bar{j}_{ext}^{\,\perp}} \quad (10)$$

$$\boxed{\varepsilon_\perp(q,\omega) = 1 + \frac{i\sigma_\perp}{\varepsilon_0\omega} = 1 + \chi + \frac{q^2c^2}{\omega^2}\chi_m} \quad (11)$$

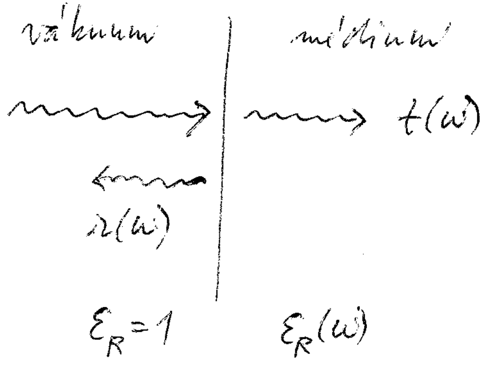
Rovnica (10) popisuje vlny $\perp$ polia.

Spontánne kmity:
$$\boxed{\omega = \frac{cq}{\sqrt{\varepsilon_\perp(q,\omega)}}}$$

Komplexný index lomu:
$$\boxed{\tilde{n} = \sqrt{\varepsilon_\perp(q,\omega)}}$$

Z rovníc (9, 11) máme
$$\boxed{\varepsilon_\parallel(0,\omega) = \varepsilon_\perp(0,\omega)}$$

(E) Meranie odrazivosti



Kolmý dopad:

$$\boxed{r(\omega) = \frac{\tilde{n}(\omega) - 1}{\tilde{n}(\omega) + 1} = \frac{n + i\varkappa - 1}{n + i\varkappa + 1}} \quad (12)$$

$$\boxed{R = r^* r = \frac{(n-1)^2 + \varkappa^2}{(n+1)^2 + \varkappa^2}}$$

$R(\omega)$ = merateľná funkcia

$$r(\omega) = \sqrt{R(\omega)}\, e^{i\theta(\omega)} \longrightarrow \boxed{\ln r(\omega) = \frac{1}{2} \ln R(\omega) + i\theta(\omega)}$$

Kramers-Kronigov vzťah (*) zo strany 3:

$$\boxed{\theta(\omega) = -\frac{1}{2\pi} P \int_0^\infty \ln\left|\frac{s+\omega}{s-\omega}\right| \frac{R'(s)}{R(s)}\, ds}$$

ak poznáme $R(\omega)$ na celej osi, potom vypočítame $\theta(\omega) \rightarrow$ poznáme $r(\omega)$

$\rightarrow$ môžeme merať $\tilde{n}(\omega)$ z rovnice (12)

Problém: $R(\omega)$ meriame iba na konečnom intervale frekvencií!

CVIČENIE (~~Hg~~ Hagen - Rubens)

Ukážme, že v kove pre $\omega \rightarrow 0$ máme $\varepsilon_R(\omega) = 1 + \frac{i\omega_p^2 \tau}{\omega}$

Nájdite asymptotické správanie $R(\omega)$ pre $\omega \rightarrow 0$ v kove.