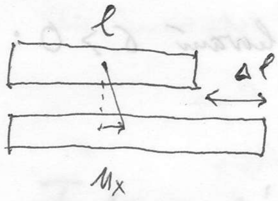
Pružné vlastnosti látok

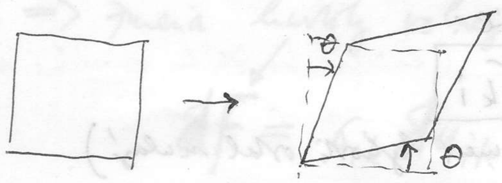
• Tenzor deformácie

bod $\vec{r} \longrightarrow \vec{r} + \vec{u}(\vec{r})$



$$u_x = \frac{\Delta l}{l}\cdot x = u_{xx}\cdot x \qquad u_{xx} = \frac{\partial u_x}{\partial x}$$

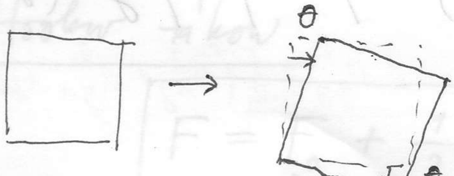
iná deformácia:



$$u_x = \theta y \qquad u_y = \theta x$$

$$u_{xy} = \frac{\partial u_x}{\partial y} = \theta \qquad u_{yx} = \frac{\partial u_y}{\partial x} = \theta$$

ale:



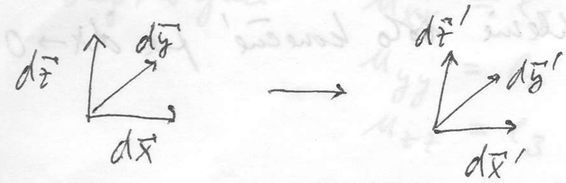
$$u_x = \theta y \qquad u_y = -\theta x$$

$$u_{xy} = -u_{xy}$$

čistá rotácia

$\Longrightarrow$ definujeme $\boxed{u_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}$ $\longrightarrow$ $u_{ij} = u_{ji}$, $u_{ij} = 0$ pre čisté rotácie

zmena objemu pri deformácii:



$$dV = dx\,dy\,dz$$

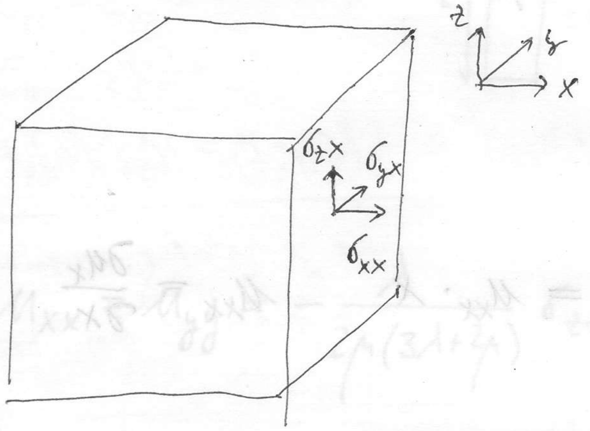
$$dV' = d\vec{x}'\cdot(d\vec{y}'\times d\vec{z}')$$

$$\frac{dV'}{dV} = \varepsilon_{\alpha\beta\gamma}\left(\delta_{x\alpha} + \partial_x u_\alpha\right)\left(\delta_{y\beta} + \partial_y u_\beta\right)\left(\delta_{z\gamma} + \partial_z u_\gamma\right)$$

$$\boxed{\frac{dV'}{dV} = 1 + \vec{\nabla}\cdot\vec{u}} + \overbrace{\left(\frac{\partial u_y}{\partial y}\frac{\partial u_z}{\partial z} - \frac{\partial u_z}{\partial y}\frac{\partial u_y}{\partial z}\right) + \left(\frac{\partial u_z}{\partial z}\frac{\partial u_x}{\partial x} - \frac{\partial u_x}{\partial z}\frac{\partial u_z}{\partial x}\right) + \left(\frac{\partial u_x}{\partial x}\frac{\partial u_y}{\partial y} - \frac{\partial u_y}{\partial x}\frac{\partial u_x}{\partial y}\right) + O(u^3)}^{\text{obyčajne zanedbávame}}$$

$V' = V + \delta V \rightarrow \boxed{\frac{\delta V}{V} = \vec{\nabla}\cdot\vec{u}}$

• Tenzor napätia $\int f_i \, dV = \int \frac{\partial \sigma_{ik}}{\partial x_k} dV = \oint \sigma_{ik} \, dS_k$



sila, ktorou pôsobí okolie elementu na element dV; pri naťahovaní $\sigma > 0$:

$$f \, dx = \sigma(x+dx) - \sigma(x)$$

Pohybová rovnica:

$$\boxed{\rho \ddot{u}_i = \frac{\partial \sigma_{ik}}{\partial x_k} + f_i}$$

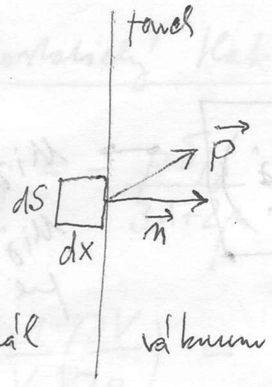
objemová sila, napr. $\vec{f} = \rho \vec{g}$ (gravitácia)

Žiadame:

$$\boxed{\sigma_{ik} = \sigma_{ki}}$$

(aby moment hybnosti ostal nulový)

Okrajová podmienka:



sila pôsobiaca na element:

$$P_i \, dS - \sigma_{ik} \, dS_k = \rho \ddot{u}_i \, dS \, dx$$

$$dS_k = n_k \, dS \qquad \vec{n}_k = \text{normála k povrchu}$$

$$\rho \ddot{u}_i = \frac{P_i - \sigma_{ik} n_k}{dx}$$

Aby zrýchlenie bolo konečné pre $dx \to 0$:

$$\boxed{P_i = \sigma_{ik} n_k}$$

vonkajší tlak; $P < 0$ pre stláčanie

príklad: hydrostatický tlak: $\sigma_{ik} = -p \, \delta_{ik} \qquad p > 0$

• termodynamika deformácie

systém koná prácu (pri infinitezimálnej deformácii $\delta u(\vec{r})$)

$$\int dV\, f_i\, \delta u_i = \int dV\, \partial_k \sigma_{ik}\, \delta u_i$$

$$= \underbrace{\int dV\, \partial_k(\sigma_{ik}\, \delta u_i)}_{=0 \text{ lebo } \delta\vec{u}=0 \text{ v nekonečne}} - \underbrace{\int dV\, \sigma_{ik}\, \partial_k \delta u_i}_{= \int dV\, \sigma_{ik}\, \delta u_{ik}}$$

Teda systém koná prácu $-\int dV\, \sigma_{ik}\, \delta u_{ik} \xrightarrow{\text{analóg}} p\,dV$

$\Rightarrow$ zmena hustoty voľnej energie

$$dF = \sigma_{ik}\, du_{ik} - S\,dT \longrightarrow \boxed{\sigma_{ik} = \left(\frac{\partial F}{\partial u_{ik}}\right)_T}$$

$$F = F(T, u_{ik})$$

• Hookov zákon

$$\boxed{F = F_0 + \frac{1}{2} C_{ij,kl}\, u_{ij}\, u_{kl}}$$

ij ... 6 hodnôt: $xx, yy, zz, xy, yz, zx \leftrightarrow 1, 2, 3, 4, 5, 6$

$C_{ij,kl}$ ... $6\times 6 = 36$ komponent

$C_{ijkl} = C_{klij} \longrightarrow 6+5+4+3+2+1 = 21$ komponent

definujme

$$\begin{aligned} u_{xx} &= e_1 & 2u_{yz} &= e_4 \\ u_{yy} &= e_2 & 2u_{zx} &= e_5 \\ u_{zz} &= e_3 & 2u_{xy} &= e_6 \end{aligned} \longrightarrow \boxed{F = F_0 + \frac{1}{2} C_{\alpha\beta}\, e_\alpha e_\beta}$$

Počet nezávislých komponent $C_{\alpha\beta}$ je diktovaný symetriou systému

Hook: $\sigma_{ij} = \left(\dfrac{\partial F}{\partial u_{ij}}\right)_T \longrightarrow \boxed{\sigma_{ij} = C_{ijkl}\, u_{kl}}$ ← faktory 2 sú OK, lebo $C_{ijkl}\underline{(u_{kl} + u_{lk})}$ faktor 2 pre $k \neq l$

$\sigma_1 = \sigma_{xx},\ \sigma_2 = \sigma_{yy},\ \sigma_3 = \sigma_{zz}$

$\sigma_4 = \sigma_{yz},\ \sigma_5 = \sigma_{zx},\ \sigma_6 = \sigma_{xy} \longrightarrow \boxed{\sigma_\alpha = C_{\alpha\beta}\, e_\beta}$

• Príklad: kubický systém

$$\left.\begin{aligned} C_{11} &= C_{22} = C_{33} \\ C_{44} &= C_{55} = C_{66} \\ C_{12} &= C_{23} = C_{31} \end{aligned}\right\} \text{3 nezávislé konštanty}$$

$C_{14} = \ldots = 0$

$\downarrow$

$C_{xxyz} = 0$

Naozaj, pri zámene $y \to -y$ sa kryštál nezmení, ale

$u_{yz} \to -u_{yz}$

$u_{xx} \to u_{xx}$

Teda

$C_{14}\, u_{xx} u_{yz} = -C_{14}\, u_{xx} u_{yz}$

(energia kryštálu musí byť rovnaká)

$\to C_{14} = 0$

$$C_{\alpha\beta} = \left(\begin{array}{ccc|ccc} C_{11} & C_{12} & C_{12} & & & \\ C_{12} & C_{11} & C_{12} & & 0 & \\ C_{12} & C_{12} & C_{11} & & & \\ \hline & & & C_{44} & 0 & 0 \\ & 0 & & 0 & C_{44} & 0 \\ & & & 0 & 0 & C_{44} \end{array}\right)$$

$$F = F_0 + \frac{1}{2} C_{11} \left(u_{xx}^2 + u_{yy}^2 + u_{zz}^2\right) + C_{12} \left(u_{xx} u_{yy} + u_{yy} u_{zz} + u_{zz} u_{xx}\right)$$
$$+ 2 C_{44} \left(u_{xy}^2 + u_{yz}^2 + u_{zx}^2\right) \quad \ldots \text{ napr. } 2 u_{xy}^2 = \frac{1}{2}\left(u_{xy}^2 + u_{yx}^2 + 2 u_{xy} u_{yx}\right)$$

alebo

$$F = F_0 + \frac{1}{2} C_{11} \left(e_1^2 + e_2^2 + e_3^2\right) + C_{12} \left(e_1 e_2 + e_2 e_3 + e_3 e_1\right) + \frac{1}{2} C_{44} \left(e_4^2 + e_5^2 + e_6^2\right)$$

$\to$ Hook: $\boxed{\begin{aligned} \sigma_{xx} &= C_{11} u_{xx} + C_{12} \left(u_{yy} + u_{zz}\right) \\ \sigma_{xy} &= 2 C_{44} u_{xy} \end{aligned}}$ a cyklické zámeny

Ak sily sú centrálne $\to$ $\boxed{C_{12} = C_{44}}$ Cauchy

v GPa

		C_{11}	C_{44}	C_{12}	
	Si	165	80	64	kovalentná väzba
	Na	7.6	4.3	6.3	kovová väzba
	Au	190	42	161	kovová väzba
(6K)	Ne	1.6	0.93	0.85	van der Waals $\approx$ Cauchy
	Fe	230	117	135	kovová väzba
(diamant)	C	1040	550	170	kovalentná väzba
	NaCl	49	13	13	iónový kryštál $\approx$ Cauchy

Príklad 2: izotropný systém → polykryštál, sklo, guma, ...

Formulu pre kubické kryštály možno písať:

$$F = F_0 + C_{44}\, u_{ij} u_{ij} + \frac{1}{2} C_{12} (u_{xx} + u_{yy} + u_{zz})^2 + \underbrace{\frac{1}{2}(C_{11} - C_{12} - 2C_{44})(u_{xx}^2 + u_{yy}^2 + u_{zz}^2)}_{\text{nie je rotačne invariantné}}$$

→ žiadame $C_{11} = C_{12} + 2C_{44}$

$$\left.\begin{array}{l} C_{44} \equiv \mu \\ C_{12} = \lambda \end{array}\right\} \text{Lamé}$$

$$\boxed{F = F_0 + \frac{\lambda}{2} u_{ii}^2 + \mu\, u_{ij} u_{ij}}$$

Hook:

$$\boxed{\sigma_{ij} = \lambda \delta_{ij}\, \vec{\nabla}\cdot\vec{u} + 2\mu\, u_{ij}}$$

$$\sigma_{ii} = (3\lambda + 2\mu)\, u_{ii}$$

$$\boxed{u_{ij} = \frac{1}{2\mu} \sigma_{ij} - \frac{\lambda \delta_{ij}}{2\mu(3\lambda + 2\mu)} \sigma_{ll}}$$

možno aj písať

$$F = F_0 + \frac{1}{2}\left(\lambda + \frac{2\mu}{3}\right) \underbrace{(u_{ii})^2}_{\text{stopa tenzora}} + \mu \left(u_{ij} - \frac{1}{3}\delta_{ij} u_{ll}\right) \underbrace{\left(u_{ij} - \frac{1}{3}\delta_{ij} u_{ll}\right)}_{\text{tenzor so stopou} = 0}$$

nezávislé veci

stabilita → $\boxed{\begin{array}{l} K = \lambda + \frac{2\mu}{3} > 0 \\ \mu > 0 \end{array}}$

pozdĺžny zvuk

$$v_{\parallel} = \sqrt{\frac{\lambda + 2\mu}{\rho}}$$

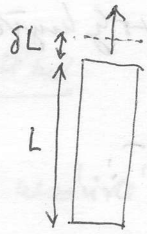
priečny zvuk

$$v_{\perp} = \sqrt{\frac{\mu}{\rho}}$$

pohybová rovnica

$$\rho \ddot{u}_i = \frac{\partial \sigma_{ij}}{\partial x_j} \longrightarrow \boxed{\rho \ddot{\vec{u}} = (\lambda + \mu) \vec{\nabla}(\vec{\nabla}\cdot\vec{u}) + \mu \Delta \vec{u}}$$

• Homogénne naťahovanie:



$\sigma_{zz} \neq 0$, ostatné $\sigma_{ij} = 0$

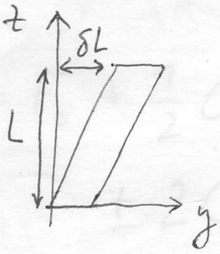
$$\sigma_{zz} = E\, u_{zz} = E\,\frac{\delta L}{L}$$

$$\boxed{E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}}$$ Youngov modul pružnosti

$$u_{xx} = u_{yy} = -\frac{\lambda}{2\mu(3\lambda+2\mu)}\,\sigma_{zz} \longrightarrow \boxed{-\frac{u_{xx}}{u_{zz}} = \frac{\lambda}{2(\lambda+\mu)} \equiv \nu}$$ Poisson

termodynamika: $\boxed{-1 < \nu < \frac{1}{2}}$

• Homogénna strihová deformácia



iba $\sigma_{yz} \neq 0$ $\longrightarrow$ $\sigma_{yz} = 2\mu\, u_{yz} = \mu\,\frac{\partial u_y}{\partial z}$

μ strihový modul

$$\boxed{\sigma_{yz} = \mu\,\frac{\delta L}{L}}$$

• Hydrostatický tlak $\sigma_{ij} = -p\,\delta_{ij}$

$$u_{ij} = -\frac{p\,\delta_{ij}}{3\lambda+2\mu} \longrightarrow \vec{\nabla}\cdot\vec{u} = -\frac{3p}{3\lambda+2\mu} \longrightarrow \frac{\delta V}{V} = \vec{\nabla}\cdot\vec{u}$$

$$-\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_T = \frac{1}{K},$$ kde $\boxed{K = \lambda + \frac{2}{3}\mu}$ modul objemovej stlačiteľnosti

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moduly izotropných látok

	E (GPa)	ν
guma	0,0005	0.5
liate olovo	5	0.5
sklo	55	0.16
liate železo	110	0.3
oceľ	200	0.3
wolfrám	400	0.3

väčšina materiálov: $\nu > 0$

auxetické materiály: $\nu < 0$

elast. konštanty: energia / objem

typické čísla: $\frac{1\,eV}{10^{-30}\,m^3} \sim 10^{11}\,\frac{N}{m^2} \sim 10^2\,GPa$

guma: iná energetická škála!