DIAGONAL SYMMETRY IN CHLADNI PLATES.

BY

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The complete solution for the nodal lines on a square plate requires both trigonometric and hyperbolic terms; but for high vibrations, the trigonometric terms will give the approximate form of the sand figures. If great accuracy is desired, the hyperbolic terms must be introduced particularly along the edges.[1] The general theorems discussed here will be developed from the simpler trigonometric forms, with the understanding that they are also true for the equations which contain both types of terms.

When the hyperbolic terms are omitted, the plate may be regarded as a square membrane with free edges. The solution for such a membrane is

$$A \cos\frac{m\pi x}{a}\cos\frac{n\pi y}{a} + B \cos\frac{n\pi x}{a}\cos\frac{m\pi y}{a} = 0, \qquad (1)$$

in which m and n are whole numbers, a is the side of the square, A and B are arbitrary numbers. The symmetrical forms are found when $A = B$. If m is odd and n even or vice versa, one diagonal of the square is a nodal line. This is shown by the following examples which are obtained from equation (1). For $m = 1$, $n = 4$, equation (1) reduces to

$$\left(\cos\frac{\pi x}{a} + \cos\frac{\pi y}{a}\right)\left(8\cos^3\frac{\pi x}{a}\cos\frac{\pi y}{a} - 8\cos^2\frac{\pi x}{a}\cos^2\frac{\pi y}{a}\right.$$
$$\left. + 8\cos\frac{\pi x}{a}\cos^3\frac{\pi y}{a} - 8\cos\frac{\pi x}{a}\cos\frac{\pi y}{a} + 1\right) = 0. \qquad (2)$$

The first factor of this equation represents the diagonal. In the equations below u is written for $\cos \pi x/a$ and v for $\cos \pi y/a$.

[1] *Phil. Mag.*, Vol. XII, Suppl. August 1931, p. 320. *Jour. Franklin Inst.* Vol. 214, No. 2, August 1932, p. 199.

The values of m and n are indicated at the extreme right.

$$(u+v)\left\{\begin{array}{l} 32u^5v - 32u^4v^2 + 32u^3v^3 - 32u^2v^4 \\ \qquad + 32uv^5 - 48u^3v + 48u^2v^2 \\ \qquad\qquad - 48uv^3 + 18uv - 1 \end{array}\right\} = 0, \quad (1, 6)$$

$$(u+v)\{u^2(8v^2-4) - 2uv - 4v^2 + 3\} = 0, \quad (2, 3)$$

$$(u+v)\left\{\begin{array}{l} u^4(32v^2 - 16) - u^3(32v^3 - 16) \\ \qquad + u^2(32v^4 - 56v^2 + 20) \\ \qquad + u(16v^3 - 10v) - 16v^4 \\ \qquad\qquad + 20v^2 - 5 \end{array}\right\} = 0, \quad (2, 5)$$

$$(u+v)\left\{\begin{array}{l} u^7(512v^3 - 384v) - u^6(512v^4 \\ \quad - 384v^2) + u^5(512v^5 - 1208v^3 \\ \quad + 768v) - u^4(512v^6 - 1208v^4 \\ \quad + 768v^2) + u^3(512v^7 - 1208v^5 \\ \quad + 1408v^3 - 480v) + u^2(384v^6 \\ \quad - 768v^4 + 352v^2) - u(384v^7 \\ \quad - 768v^5 + 480v^3 + 92v) + 4v^2 - 3 \end{array}\right\} = 0. \quad (3, 8)$$

All of these equations have the factor $(u + v) = 0$ which represents one straight line diagonal. In the experimental production of these lines, it is extremely difficult to keep A exactly equal to B. This causes the diagonal to vary slightly from a straight line (Fig. 1*a* and 1*b*).

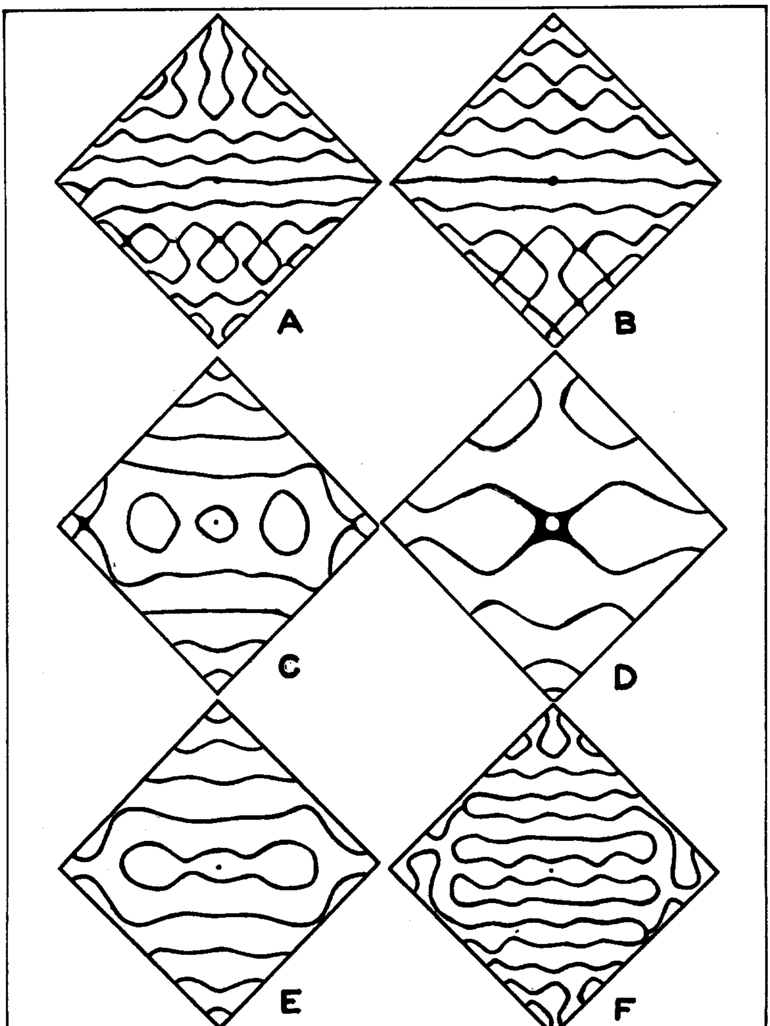
There is also the possibility of a second series of nodal lines in which the diagonal is one of maximum instead of minimum vibration. This can be visualized physically by supposing that the diagonal divides the square into two similar triangles each vibrating in such a way that along the common edge (the diagonal) the vibrations either annul one another to form a nodal line (Fig. 1*a* and 1*b*) or reinforce one another to form a line of maximum vibration (Fig. 1*c*, 1*d*, 1*e*, 1*f*).

As m and n are increased in value, the note rises higher and higher. The nodal lines become greater in number and the resulting figures become more complicated. Such a series is shown in Fig. 2.

The two plates in Fig. 3 have a peculiar symmetry which is not accounted for by the theory given above. If each plate is divided into two equal triangles by a diagonal and one triangle rotated through 180° it will fit over the other triangle.

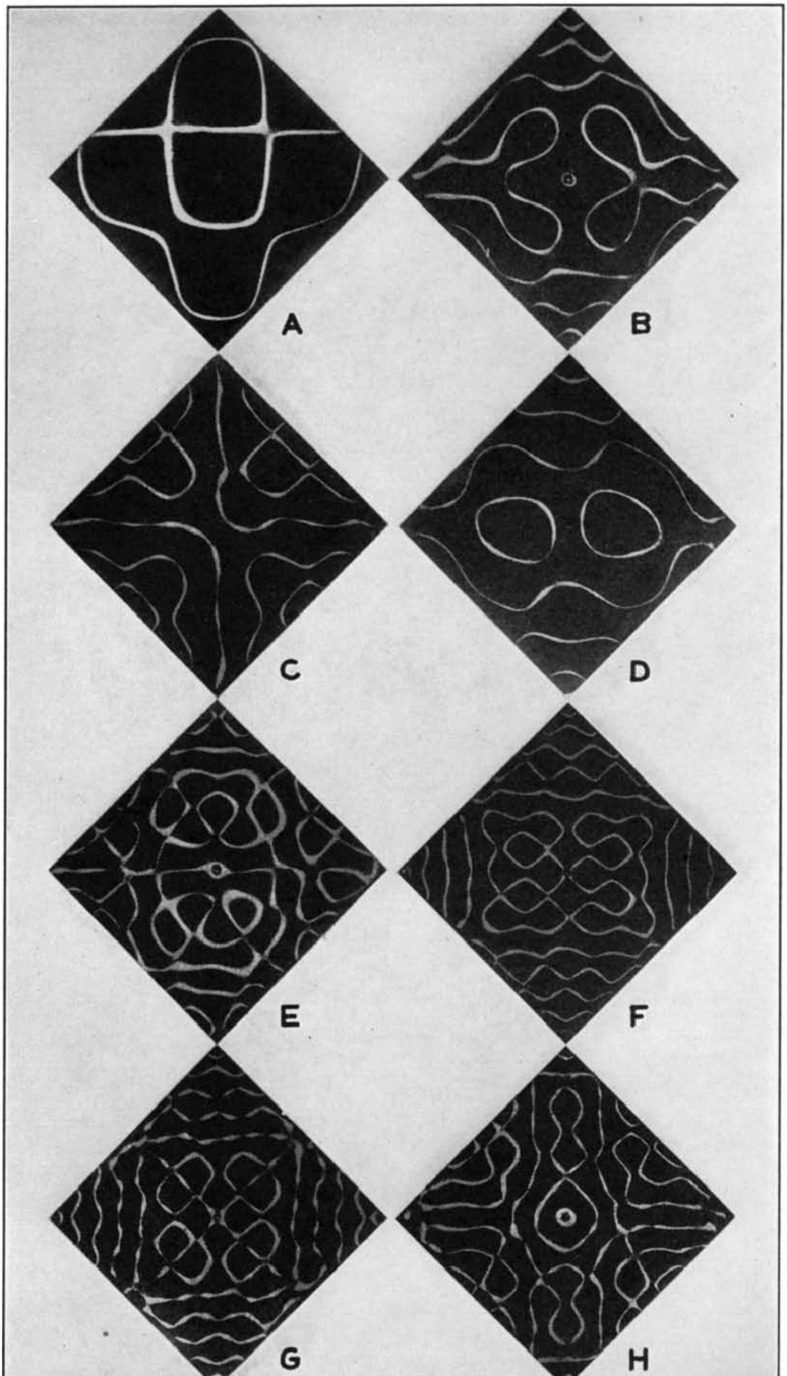
The series of plates in Fig. 4 shows how a slight change in A and B carries the nodal lines over from symmetry about a single diagonal in Fig. 4*a* to symmetry about both diagonals and both median lines in Fig. 4*f*. So far as the writer knows,

FIG. 1.



These figures are tracings from photographic negatives. In *a* and *b* one diagonal is a nodal line; in *c*, *d*, *e*, and *f* the corresponding diagonal is a line of maximum vibration.

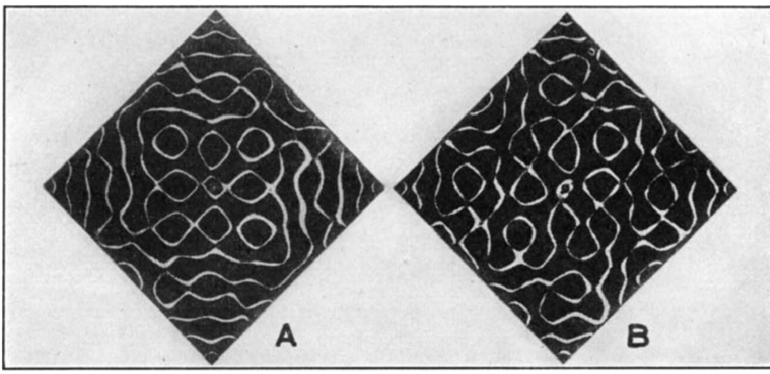
FIG. 2.



The nodal lines increase as the note of the oscillator rises in pitch. The increasing complexity of the figures is shown from *a* to *h*.

a series like this has never been found before experimentally but a theory of such a transformation has been given by Ritz.[2] When m and n are odd, both diagonals are nodal lines as well as the two median lines passing through the center of the square. If m and n are even numbers and multiples of odd and even numbers respectively, the resultant curves have nodal diagonals but the median lines are not nodal. These

FIG. 3.



These nodal patterns show a new type of symmetry.

are illustrated in Fig. 5 with photographic reproductions of the Chladni plate. The theoretical curves plotted from the equation,

$$\cos \frac{m\pi x}{a} \cos \frac{n\pi y}{a} - \cos \frac{n\pi x}{a} \cos \frac{m\pi y}{a} = 0, \tag{3}$$

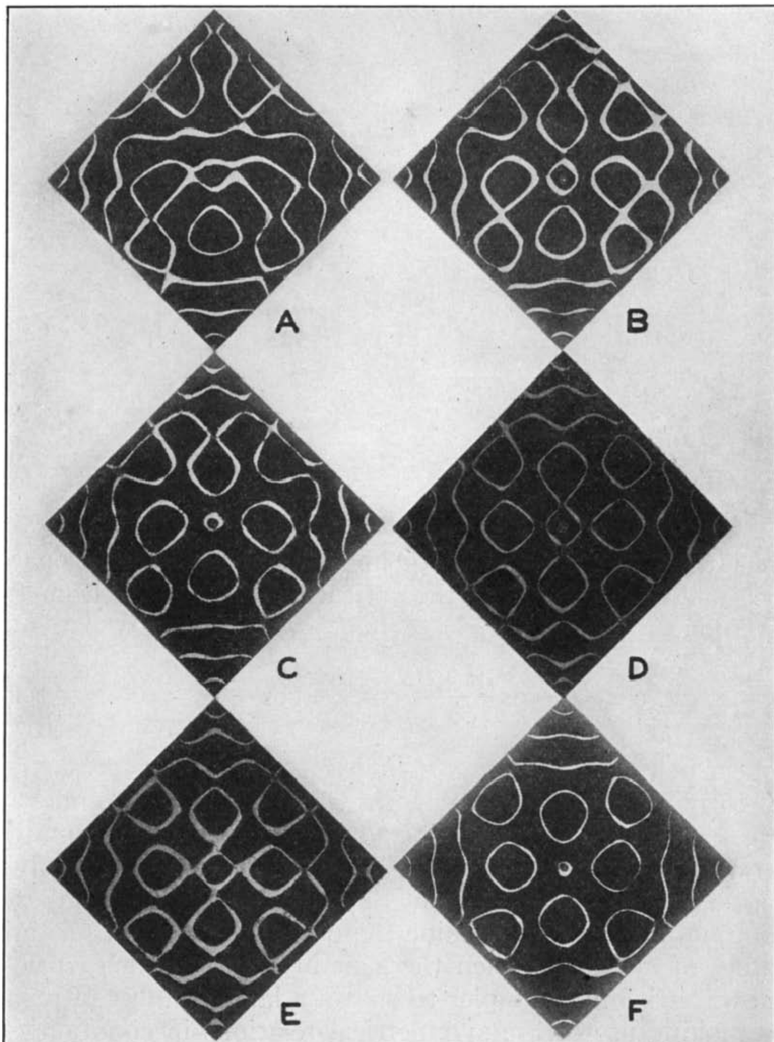
appear in Fig. 6.

In plotting the curves of the Chladni plates one may reduce equation (3) to equations of the form given in equation (2) and those similar to it. These equations are then solved approximately by Horner's method and a sufficient number of points obtained to determine the curve. However, for large values of m and n, when the general shape of each curve is known, it is much simpler to obtain a large number of points by making use of the symmetrical relations in equation (3).

[2] Ritz, *Ann. der. Phy.*, Vol. 28, 1909, page 770.

Thus for the group shown in Fig. 6, the relation $n = m + 2$ is true for each nodal system shown. All the figures are symmetrical about the center and the four corners; the diagonal lines are lines of no vibration. Hence x and y are inter-

FIG. 4.



The nodal pattern assumes many forms for a single note when *A* and *B* are varied.

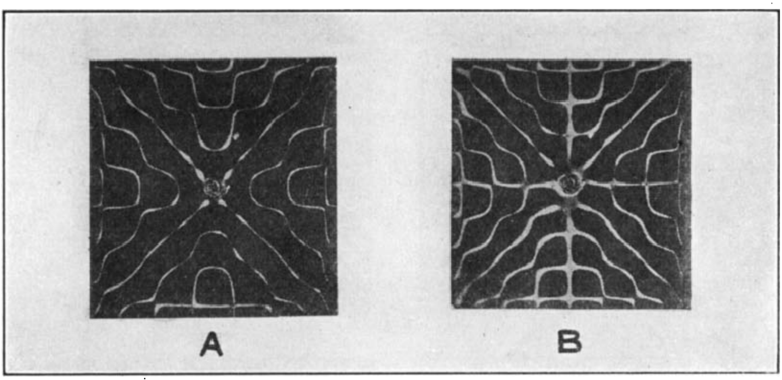
changeable. This means that for any values of x, there are corresponding values of y; more specifically, if

$$x = \frac{N_0 a}{2m},$$

then

$$y = \frac{N_0 a}{2m}, \qquad (4)$$

FIG. 5.



These experimental curves should be compared with the theoretical curves given in Fig. 6.

in which N_0 is any odd number between 0 and $2m$. Also if

$$x = \frac{N_0 a}{2n},$$

then

$$y = \frac{N_0 a}{2n}. \qquad (5)$$

Similarly if

$$x = \frac{N_0 a}{m + n},$$

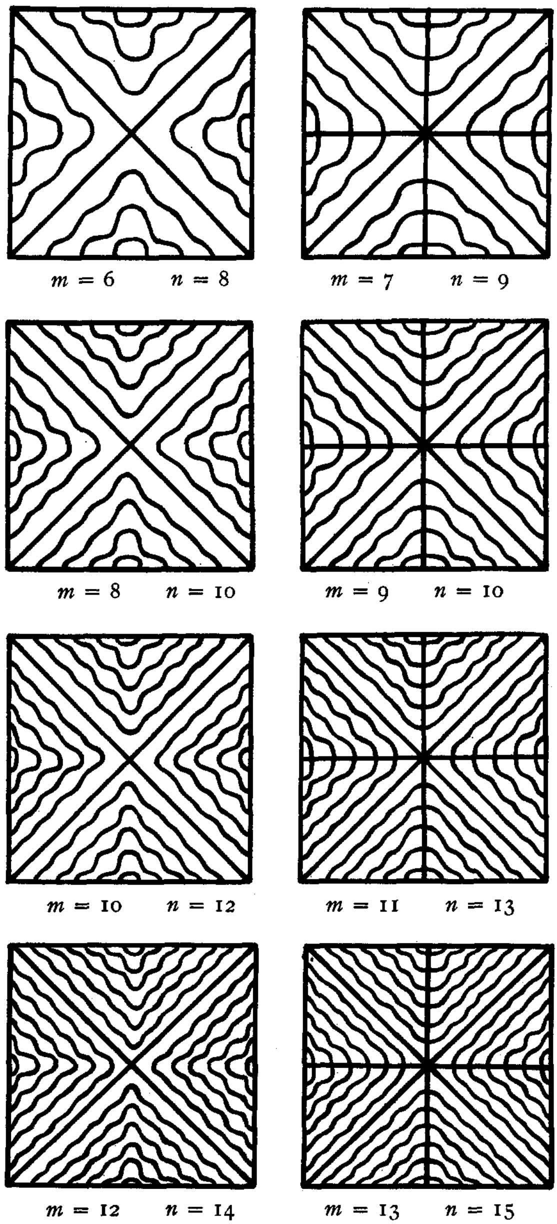
then

$$y = \frac{N_0 a}{m + n}, \qquad (6)$$

and if

$$x = \frac{N_e a}{m + n},$$

FIG. 6.



These curves are worked out from equation (1). They are symmetrical about both diagonals and both median lines through the center parallel to the edges. Those on the left have anti-nodal median lines; those on the right have nodal medians.

then

$$y = \frac{N_e a}{m + n}, \tag{7}$$

in which N_e is any even number between 0 and $(m + n)$. For the values $m = 9$, $n = 11$, we obtain from equation (4) the values

$$\frac{x}{a} = \frac{1}{18}, \frac{3}{18}, \frac{5}{18}, \frac{7}{18}, \frac{9}{18}, \frac{11}{18}, \frac{13}{18}, \frac{15}{18}, \frac{17}{18}, \cdots \tag{8}$$

with exactly similar values for y/a. These two sets of values are true in every case, so they give $9 \times 9 = 81$ points on the required curves.

The values of N_0 in equation (5) give

$$\frac{x}{a} = \frac{1}{22}, \frac{3}{22}, \frac{5}{22}, \frac{7}{22}, \cdots \frac{21}{22}$$

or eleven points in all. There are eleven corresponding points for y/a; so that the possible combinations give $11 \times 11 = 121$ points. The odd numbers in (6) give

$$\frac{x}{a} = \frac{1}{20}, \frac{3}{20}, \frac{5}{20}, \frac{7}{20}, \frac{9}{20}, \frac{11}{20}, \frac{13}{20}, \frac{15}{20}, \frac{17}{20}, \frac{19}{20}$$

with ten similar values for y/a. We thus obtain $10 \times 10 = 100$ points from this equation and 100 more from equation (7). This gives a total of 402 points on the curves. The manner in which these points are to be connected together is determined by the family to which the curves belong. This family is found by plotting the first few curves with Horner's method.

The theory given in this article relates every nodal figure on a Chladni plate to a very similar figure upon a supposititious membrane with free edges. The differential equation of the plate is of the fourth degree, while the membrane has a second degree equation. It is thus possible to find an approximate solution for the plate using only trigonometric functions. These theoretical figures can be modified along the edges by the introduction of hyperbolic functions and they will then agree with the experimental curves, the error being approximately two per cent. The sand lines were produced with the valve oscillator described in the papers to which reference is made.