The Calculation of Chladni Patterns

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AN APPROXIMATE solution for the Chladni patterns on square plates has been developed in the form

$$W = A \cos\frac{m\pi x}{a}\cos\frac{n\pi y}{a} - B\cos\frac{n\pi x}{a}\cos\frac{m\pi y}{a} = 0. \quad (1)$$

Either term of this equation represents straight lines parallel to the sides of the square, but when the two terms are mixed in any desired proportion by giving the proper values to A and B, all the varied nodal curves may be worked out. Since m and n are interchangeable in the two terms, both terms may be formed at the same frequency because the frequency is also dependent upon m and n. In the case of a circular plate (or membrane) a fairly accurate solution is given by the equation

$$W = AJ_n(kr)\cos n(\theta-\alpha_n) - BJ_m(k'r)\cos m(\theta-\alpha_m) = 0. \quad (2)$$

The first term of this equation $AJ_n(kr) \times\cos n(\theta-\alpha_n)$ has heretofore been given as the complete solution (Kirchhoff's) of a vibrating plate. In reality, however, it represents only circles and radii corresponding to the straight lines on the square plate. If a second vibration, which may arise in many ways, is represented by the second term $BJ_m(k'r)\cos m(\theta-\alpha_m)$, then these two may be added together with different values of A and B to represent the complicated systems of nodal lines which appear on circular plates. There is this difference between the square plates and the circular ones. Every pattern which appears on a square plate has its mathematical equation and every pattern calculated from Eq. (1) can be produced upon a square plate. In the circular plates every experimentally produced nodal system must have an appropriate mathematical formula; but many beautiful figures which occur in the mathematical equations cannot be produced upon the circular plates, the reason being that the k and k' of Eq. (2) cannot be produced together in every case.

In order to work out the nodal patterns for circular plates from Eq. (2), it is better to adopt a graphical method. For convenience the two tables printed in a previous paper[1] are repeated here. Table I gives the value of ka for various nodal systems. Poisson's ratio is assumed to be 0.33 and a is the radius of the plate. Eq. (2) is first thrown into the form

$$y = J_m(k'r), \quad y = KJ_n(kr),$$

where
$$K = \frac{A\cos n(\theta-\alpha_n)}{B\cos m(\theta-\alpha_m)}. \quad (3)$$

The required lines consist of two basic systems of the Kirchhoff type added together in certain proportions determined by K. For example let the curve be intermediate between the Kirchhoff form of one circle and two diameters, and that containing two circles and no diameters, or mathematically $S=1$, $n=2$ and $S=2$, $n=0$. From Table I, the values for ka are 5.937 and

TABLE I. *Values of ka for various nodal systems. n is the number of diameters; S the number of circles.*

S	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$
0	..	..	2.292*	3.497	4.65†	5.75†	6.80†
1	3.014	4.530*	5.937*	7.274	8.55†	9.79†	11.00†
2	6.209	7.737*	9.16	10.55	11.95	13.23	14.50
3	9.370	10.91*	12.41	13.86	15.24	16.57	17.88
4	12.53	14.08	15.58	17.05	18.45	19.81	21.15
5	15.68	17.23	18.73	20.21	21.63	23.01	24.37
6	18.83	20.38	21.89	23.37	24.80	26.20	27.57
7	21.98	25.53	25.04	26.52	27.96	29.40	30.86
8	25.12	26.67	28.19	29.67	31.12	32.58	34.04
9	28.26	29.81	31.33	32.81	34.28	35.74	37.21
10	31.40	32.95	34.47	35.95	37.43	38.90	40.38

* Values also given by Timoshenko (*Vibration Problems in Engineering* (1928), p. 317).
† Values true within 2 percent.

[1] R. C. Colwell and H. C. Hardy, Phil. Mag. **24**, 1041 (1937).

6.209, respectively. Hence Eq. (2) becomes

$$AJ_2\left(\frac{5.937r}{a}\right)\cos 2\theta - BJ_0\left(\frac{6.209r}{a}\right)=0. \quad (4)$$

Now by making A large as compared to B, the curve approximates the Kirchhoff type of one circle and two diameters. By making B much larger than A, the curve approaches two circles. If we take $A=5$ and $B=1$, then

$K=5\cos 2\theta$,

$$y=KJ_2\left(\frac{5.937r}{a}\right), \quad y=J_0\left(\frac{6.209r}{a}\right). \quad (5)$$

The period of K is π and the curve is symmetric with respect to $\theta=0$ and $\theta=\pi/2$. It is, therefore, only necessary to plot the curve from θ to $\pi/2$ and obtain the remainder by reflections. The graphs of Eq. (5) for $K=1$ and $K=5$ are shown in Fig. 1. The value of K is easily found for any value of θ. Then the plot of the second two equations in (5) on axes y and r/a gives at their intersection the values of r/a corresponding to θ. With the change in θ, the values of K are found which change the amplitude of J_2. The intersections of $y=J_0$ and $y=KJ_2$ are read from the

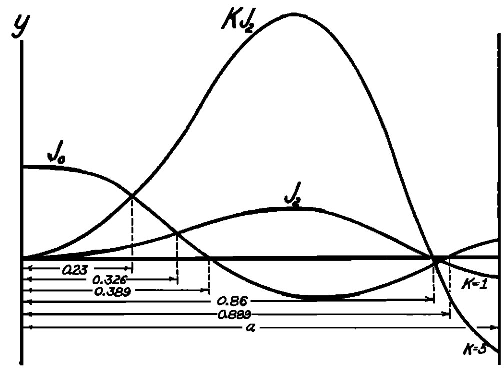


FIG. 1.

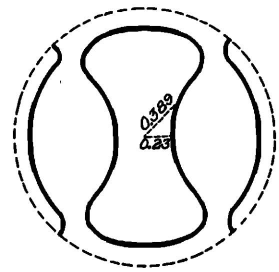


FIG. 2.

curves with a fair degree of accuracy. These values are plotted against θ in polar coordinates. A tabulation of the results for Eq. (5) gives Table III. From this table it is apparent that one branch of the curve advances from $r=0.889a$ at $\theta=45°$ to $r=a$ before θ reaches 50°. When r is plotted against θ, the curve of Fig. 2 results. The shape of the curve may be changed from one form to another by changing the ratio of A to B.

If the Kirchhoff form of two circles and two diameters is to be changed to another Kirchhoff

TABLE II. *Roots of* $J_n(x)=0$.

S	n=0	n=1	n=2	n=3	n=4	n=5
1	2.405	3.832	5.135	6.379	7.586	8.780
2	5.520	7.016	8.417	9.760	11.064	12.339
3	8.624	10.173	11.620	13.017	14.373	15.700
4	11.792	13.323	14.796	16.224	17.616	18.982
5	14.931	16.470	17.960	19.410	20.827	22.220
6	18.071	19.616	21.117	22.583	24.018	25.431
7	21.212	22.760	24.270	25.749	27.200	28.628
8	24.353	25.903	27.421	28.909	30.371	31.813
9	27.494	29.047	30.571	32.050	33.512	34.983
10	30.636	32.160	33.720	35.192	36.654	38.148

TABLE III.

θ	K	r/a		θ	K	r/a
0	5.000	0.23	0.87	50	− .868	0.65
5	4.924	0.235	0.87	55	−1.710	0.83
10	4.698	0.241	0.871	60	−2.50	0.84
15	4.330	0.250	0.872	65	−3.214	0.844
20	3.830	0.261	0.872	70	−3.830	0.848
25	3.214	0.27	0.873	75	−4.330	0.852
30	2.500	0.28	0.874	80	−4.698	0.855
35	1.710	0.306	0.876	85	−4.924	0.858
40	0.868	0.34	0.882	90	−5.000	0.86
45	0.	0.387	0.889			

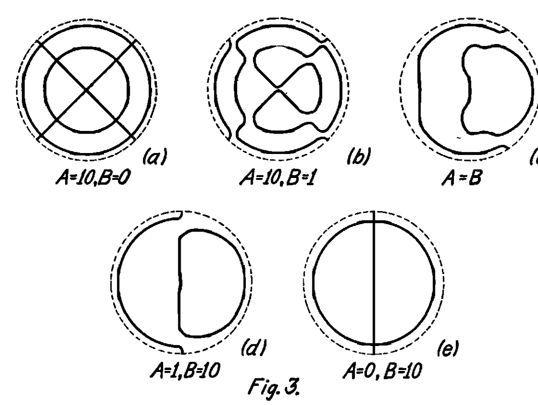
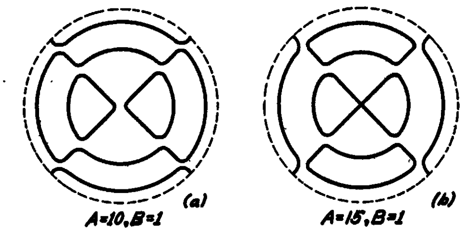


FIG. 3.



(a) (b)

FIG. 4.

form involving only one circle and one diameter, the ratio of A to B must be varied from infinity to zero. The steps are shown in Fig. 3 as the ratio varies from infinity through 10, 1, 1/10 and 0. The equation is

$$AJ_2\left(\frac{9.16r}{a}\right)\cos 2\theta - BJ_1\left(\frac{4.35r}{a}\right)\cos\theta = 0. \quad (6)$$

Approximate solutions are also obtainable in which the Bessel functions are replaced by trigonometric functions only. It should be understood, however, that although the patterns found are similar in appearance to the experimental curves, they are somewhat distorted and are not correctly situated on the plate. The first approximation comes from the observation that the curve $y=J_n(x)$ has a nearly sinusoidal form except that the first periods are stretched in the neighborhood of the origin and the amplitude decreases as x increases. The curve $y=J_n(x)$ may be replaced by $y=ae^{-bx}\cos mx$. The value

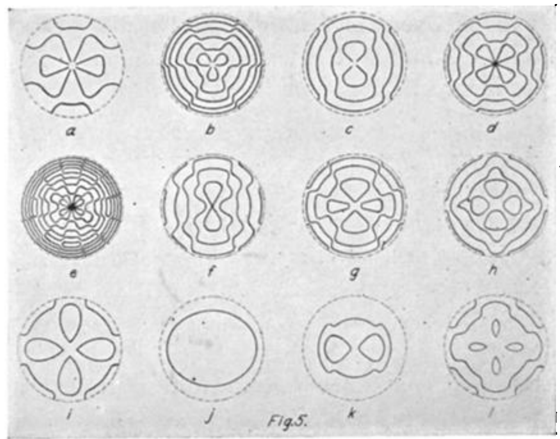


FIG. 5.

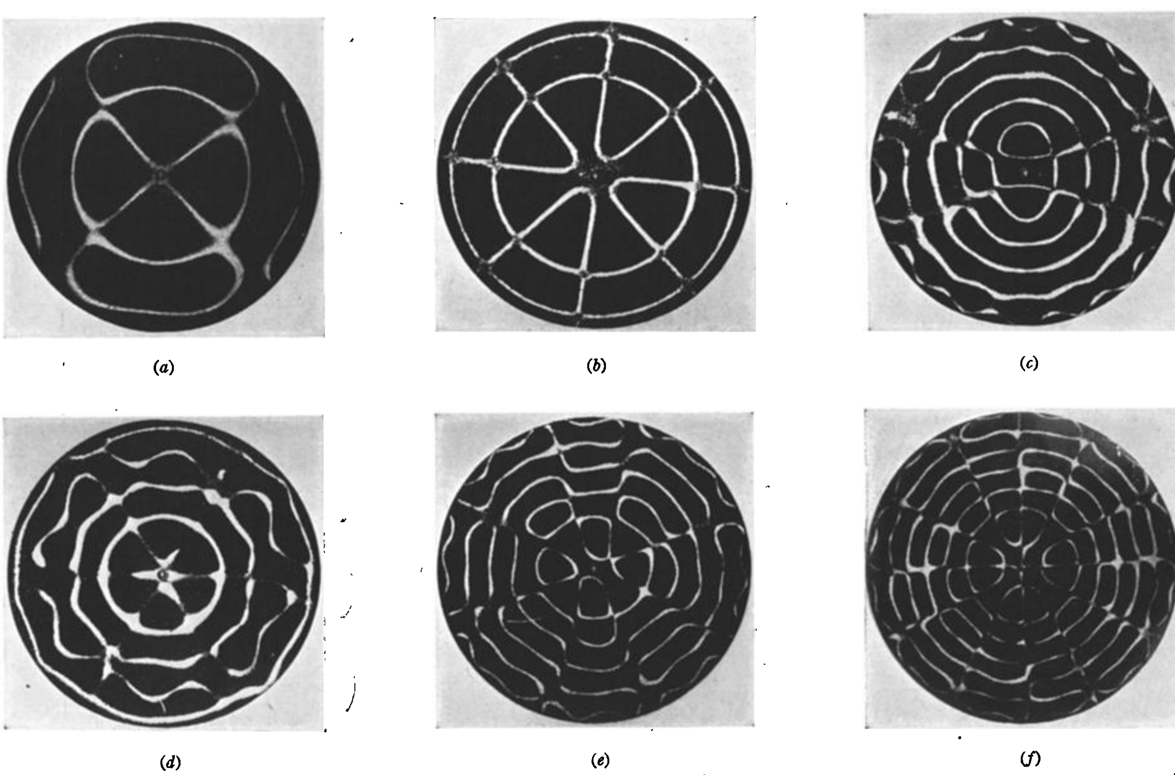


FIG. 6.

of the exponential factor may be disregarded when the number of circles in the two ideal base types is small. Hence we have

$$y=\cos\frac{nr}{a},\quad y=K\cos\frac{mr}{a},\quad K=\frac{A\cos n'\theta}{B\cos m'\theta},\quad (7)$$

where the resulting curve is intermediate between one with n circles and n' diameters and one with m circles and m' diameters. It must be kept in mind, however, that the Bessel functions do not behave as cosine functions at zero, so that the resulting patterns are inaccurate near the center of the plate.

It is also possible that the patterns found are made up of a number of Bessel functions such that

$$y=\sum_1^n J_n x.$$

An appropriate approximation would then be[2]

$$\cos x=J_0(x)-2J_2(x)-2J_4(x),\quad (8)$$

which is similar to Eq. (7). These equations depend upon the first power of r/a; but there is no reason for thinking that higher powers do not enter the equation. For instance let us suppose that in some way all vibrations are suppressed except these due to $(r/a)^2$. Let us then take an example in which $m=n$. It follows that the two auxiliary curves (Eq. (7)) are out of phase since $J_n(x)$ and $J_m(x)$ behave in this way. Then if the two base curves are respectively two circles with two diameters and two circles only, we write

$$A\cos\frac{2\pi r^2}{a^2}\cos 2\theta-B\cos\left(\frac{2\pi r^2}{a^2}+\frac{\pi}{2}\right)=0,\quad (9)$$

or

$$A\cos\frac{2\pi r^2}{a^2}\cos 2\theta-B\sin\frac{2\pi r^2}{a^2}=0.\quad (10)$$

The graph of this equation appears in Fig. 4 (b). A similar pattern has been found experimentally. The pattern Fig. 4 (a) comes from the equation

$$10J_2\left(\frac{9.16r}{a}\right)\cos 2\theta-J_0\left(\frac{3.014r}{a}\right)=0.\quad (11)$$

The base pattern for Eq. (11) consists of one curve with two circles and two diameters; the other curve has one circle only.

In conclusion a few other patterns will be given and the equations from which they were plotted (Eqs. (12), Fig. 5).

$$\left.\begin{aligned}
&5\cos\frac{\pi r^2}{a^2}\cos 4\theta+\cos\frac{2\pi r^2}{a^2}\cos 2\theta=0 && (5a)\\
&10\cos\frac{5\pi r^2}{a^2}\cos 3\theta+\cos\frac{4\pi r^2}{a^2}=0 && (5b)\\
&10\cos\frac{3\pi r^2}{a^2}\cos 2\theta-\cos\frac{2\pi r^2}{a^2}=0 && (5c)\\
&10\cos\frac{3\pi r^2}{a^2}\cos 4\theta-\cos\frac{2\pi r^2}{a^2}=0 && (5d)\\
&10\cos\frac{8\pi r^2}{a^2}\cos 6\theta-\cos\frac{6\pi r^2}{a^2}\cos 8\theta=0 && (5e)\\
&10J_4\left(\frac{15.24r}{a}\right)\cos 4\theta-J_2\left(\frac{15.58r}{a}\right)\cos 2\theta=0 && (5f)\\
&10J_5\left(\frac{16.57r}{a}\right)\cos 4\theta-J_1\left(\frac{17.23r}{a}\right)\cos\theta=0 && (5g)\\
&J_5\left(\frac{16.57r}{a}\right)\cos 5\theta-J_1\left(\frac{17.23r}{a}\right)\cos\theta=0 && (5h)\\
&J_0\left(\frac{9.37r}{a}\right)-10J_4\left(\frac{8.55r}{a}\right)\cos 4\theta=0 && (5i)\\
&J_0\left(\frac{3.014r}{a}\right)-J_2\left(\frac{9.16r}{a}\right)\cos 2\theta=0 && (5j)\\
&10J_0\left(\frac{3.014r}{a}\right)-J_2\left(\frac{9.16r}{a}\right)\cos 2\theta=0 && (5k)\\
&10J_4\left(\frac{11.95r}{a}\right)\cos 4\theta-J_0\left(\frac{3.014r}{a}\right)=0. && (5l)
\end{aligned}\right\}(12)$$

[2] A. Gray, G. B. Matthews and T. M. MacRobert, *Bessel Functions* (Macmillan Co., 1922), p. 42.

The equations in this group which contain Bessel functions represent the usual two normal components of a free vibration. They can be found experimentally. The others represented by cosine functions only will appear when suitable constraints are applied to the plate. We have not been able to discover how the constraints should be applied in any given case but we have been able to produce some of the patterns.

The photographic reproductions (Fig. 6) show clearly that two normal vibrations may occur at the same time on a circular plate, and thus produce symmetrical figures which are neither diameters nor circles.

Our thanks are due to the Western Electric Company which donated the audio-oscillator used in producing the patterns of Fig. 6.