Introduction to fractional laplacian

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Applications of fractional Laplacian

Fractional Laplacian is a mathematical tool to describe anomalous diffusion. Some applications of fractional Laplacian include:

- diffusion on porous media is accelerated by capillary action
- cellular biology and active transport
- financial derivatives pricing, when underlying asset price can "jump"
- turbulence and non-linear flows
- population migration in epidemiological model

Ordinary diffusion – discrete case

Consider a diffusion problem on a line. There are many small cubes, each with different concentration of certain substance. Two neighbouring compartments are connected, so substance can flow between them:

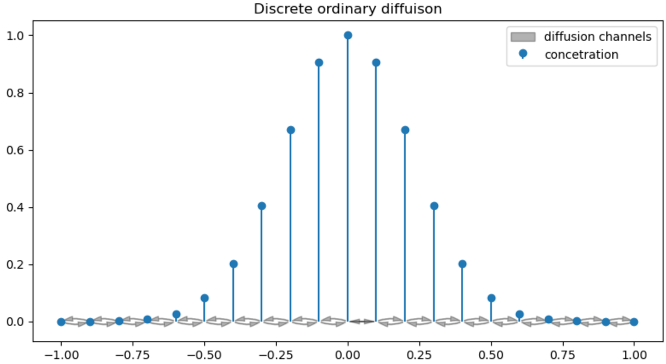


Fig. 1. Ordinary diffusion, discrete version

Ordinary diffusion – discrete case

Let us denote concentration at compartment x at time t as $u(t,x)$.
The rate of change in concentration in compartment x can be modeled as:

$$a^2 C_{dx}[\underbrace{u(t,x-dx)-u(t,x)}_{\text{net flow to the left}}+\underbrace{u(t,x+dx)-u(t,x)}_{\text{net flow to the right}}]$$
$$= a^2 C_{dx}[u(t,x-dx)-2u(t,x)+u(t,x+dx)],$$

where a is diffusion coefficient (the bigger a, the faster diffusion) and dx is distance between two neighboring compartments and C_{dx} is normalization constant to ensure consistency as dx changes.

Now, taking limit $dx \to 0^+$, we get the rate of change of concentration is

$$a^2 \frac{\partial^2 u(t,x)}{\partial x^2}$$

Parabolic partial differential equations

To describe standard diffusion, we use parabolic partial differential equation. In case of one spatial variable x, the equation has form:

$$\frac{\partial u(t,x)}{\partial t} = a^2 \frac{\partial^2 u(t,x)}{\partial x^2} = a^2 \Delta u(t,x)$$

with some initial condition $u(t,x)|_{t=0} = u_0(x)$. Parameter a is called *diffusion coefficient*. Operator Δ is called *Laplace operator*.

Note that diffusion rate depends on second partial derivative, which is local operator: it can be calculated from arbitrary small neighborhood of x, e.g. via limit:

$$\frac{\partial^2 u(t,x)}{\partial x^2} = \lim_{h \to 0} \frac{u(t,x-h) - 2u(t,x) + u(t,x+h)}{h^2}.$$

Cosine initial condition

Let us solve diffusion equation with cosine initial condition:

$$\frac{\partial u(t,x)}{\partial t} = a^2 \frac{\partial^2 u(t,x)}{\partial x^2}, \quad u_0(x) = A_\omega \cdot \cos(\omega x)$$

One might expect a solution that is just scaled cosine:

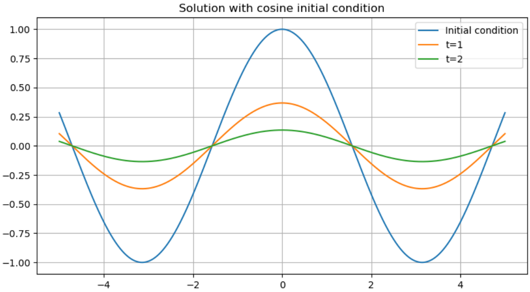


Fig. 2. Solution to diffusion equation with cosine initial condition

Cosine initial condition

Formally, suppose that solution is really just scaled cosine function:

$$u(t,x) = f(t)\cos(\omega x)$$

Then, its partial derivatives are:

$$\begin{aligned}\frac{\partial u(t,x)}{\partial t} &= \frac{df(t)}{dt}\cos(\omega x)\\ \frac{\partial^2 u(t,x)}{\partial x^2} &= -f(t)\omega^2\cos(\omega x)\end{aligned}$$

Substituting these into original equation yields ordinary differential equation for f:

$$\frac{df(t)}{dt}\cos(\omega x) = -f(t)\cdot a^2\omega^2\cos(\omega x)$$

Cosine initial condition

The equation for scaling function f can be solved explicitly:

$$\begin{aligned}\frac{df(t)}{dt}\cos(\omega x) &= -f(t)\cdot a^2\omega^2\cos(\omega x) \quad / : \cos(\omega x)\\ \frac{df(t)}{dt} &= -f(t)\cdot a^2\omega^2\\ f(t) &= f(0)\exp\left(-a^2\omega^2 t\right)\end{aligned}$$

with $f(0) = A_\omega$, to match initial condition $u(0,x) = A_\omega \cos(\omega x)$.
Therefore a solution to diffusion equation is:

$$u(t,x) = A_\omega \exp\left(-a^2\omega^2 t\right)\cos(\omega x)$$

Fourier series

The diffusion equation is *linear*. If solution for initial condition $u_0(x) = A_\omega \cos(\omega x)$ has form:

$$u(t,x) = A_\omega \exp\left(-a^2\omega^2 t\right)\cos(\omega x),$$

then solution for initial condition: $u_0(x) = \sum_i A_{\omega_i} \cos(\omega_i x)$ has form:

$$u(t,x) = \sum_i A_{\omega_i} \exp\left(-a^2\omega_i^2 t\right)\cos(\omega_i x).$$

Note on central limit theorem

Consider any displacement function with finite variance, e.g. substance can diffuse up to 50 compartments to the left or to the right. To approximate distribution after a long time, we can use central limit theorem:

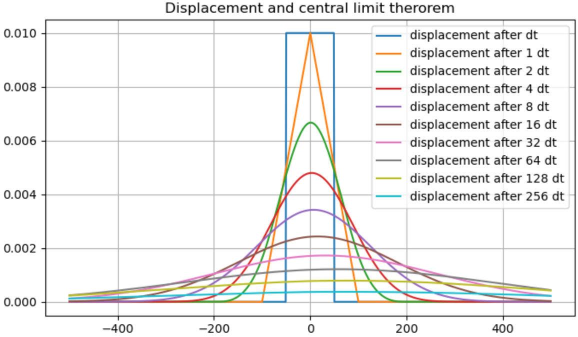


Fig. 3. Displacement function approaches Gaussian distribution

Anomalous diffusion – discrete case

Let us consider a discrete diffusion problem. We discretize x-axis into small interval of length dx. However, imagine that particles are so energized, that they can jump not only to any compartments:

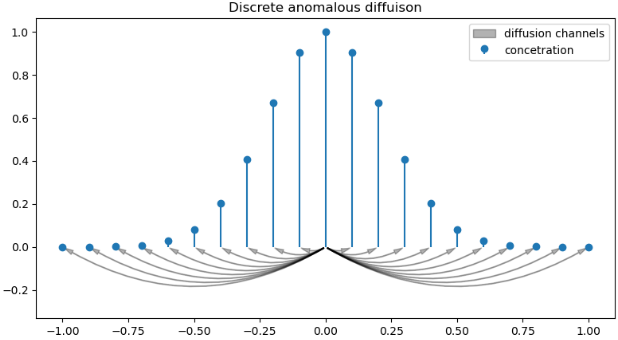


Fig. 4. Anomalous diffusion, discrete version

Power-law jump rates

We can model diffusion rates between x and $x + s \cdot dx$ as governed by power-law:

$$a^2 C_{dx,\alpha} \frac{u(t, x + s \cdot dx) - u(t, x)}{|s \cdot dx|^{1+\alpha}}$$

where a is diffusion coefficient, $\alpha \in (0, 2]$ is describing how far can particles jump and $C_{dx,\alpha}$ is a normalization constant.
Then, total rate of change in concentration at compartment x is:

$$C_{dx,\alpha} \sum_{s \neq 0} \frac{u(t, x + s \cdot dx) - u(t, x)}{|s \cdot dx|^{1+\alpha}}$$

As compartment size gets infinitesimally small, we get to integral:

$$c_{1,\alpha} \text{ p.v.} \int_{-\infty}^{\infty} \frac{u(t, x + s) - u(t, x)}{|s|^{1+\alpha}} ds,$$

which we denote as $\Delta^{\alpha/2} u(t, x)$

Fractional parabolic partial differential equations

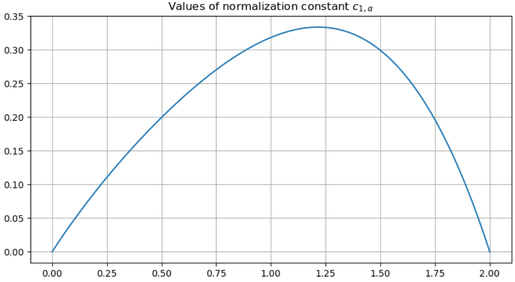
To describe non-standard diffusion, we use parabolic partial differential equation. In case of one spatial variable x, the equation has form:

$$\frac{\partial u(t,x)}{\partial t} = a^2 \Delta^{\alpha/2} u(t,x)$$

with some initial condition $u(t,x)|_{t=0} = u_0(x)$.

Parameter $c_{1,\alpha}$ is *scaling constant* with value

$$c_{1,\alpha} = \frac{-1}{2\Gamma(-\alpha)\cos\left(\frac{\alpha\pi}{2}\right)}$$



Note on principal value

Riemann integral would diverge for $s \to 0$. That's why we need to have "principal value" integral, i.e. integral everywhere but small (symmetric) neighborhood of $s = 0$:

$$\text{p.v.}\int_{-\infty}^{\infty} \frac{u(t,x+s)-u(t,x)}{|s|^{1+\alpha}} ds =$$

$$\lim_{\varepsilon \to 0^+} \left[\int_{-\infty}^{-\varepsilon} \frac{u(t,x+s)-u(t,x)}{|s|^{1+\alpha}} ds + \int_{\varepsilon}^{\infty} \frac{u(t,x+s)-u(t,x)}{|s|^{1+\alpha}} ds \right]$$

This is analogous to sum through $s \neq 0$.

Integral regularization

One can manipulate the principal value integral to get standard improper integral:

$$\lim_{\varepsilon\to 0^+}\left[\int_{-\infty}^{-\varepsilon}\frac{u(t,x+s)-u(t,x)}{|s|^{1+\alpha}}ds+\int_{\varepsilon}^{\infty}\frac{u(t,x+s)-u(t,x)}{|s|^{1+\alpha}}ds\right]$$

$$=\lim_{\varepsilon\to 0^+}\left[\int_{\varepsilon}^{\infty}\frac{u(t,x-s)-u(t,x)}{|s|^{1+\alpha}}ds+\int_{\varepsilon}^{\infty}\frac{u(t,x+s)-u(t,x)}{|s|^{1+\alpha}}ds\right]$$

$$=\lim_{\varepsilon\to 0^+}\left[\int_{\varepsilon}^{\infty}\frac{u(t,x-s)-2u(t,x)+u(t,x+s)}{|s|^{1+\alpha}}ds\right]$$

$$=\int_{0}^{\infty}\frac{u(t,x-s)-2u(t,x)+u(t,x+s)}{s^{1+\alpha}}ds$$

Cosine initial condition

Again, consider initial condition $u_0(x) = A_\omega \cos(\omega x)$. Our guess is that solution might be still separable:

$$u(t,x) = f(t)\cos(\omega x)$$

Partial derivative with respect to time is:

$$\frac{\partial u(t,x)}{\partial t} = \frac{df(t)}{dt}\cos(\omega t)$$

Fractional Laplacian of the function is a little harder.

Cosine initial condition

$$
\begin{aligned}
&\Delta^{\alpha/2}u(t,x)\\
=&\int_0^\infty \frac{f(t)\cos(\omega(x-s)) - 2f(t)\cos(\omega x) + f(t)\cos(\omega(x+s))}{s^{1+\alpha}}ds\\
=&f(t)\int_0^\infty \frac{\cos(\omega(x-s)) - 2\cos(\omega x) + \cos(\omega(x+s))}{s^{1+\alpha}}ds
\end{aligned}
$$

Manipulating numerator of function inside integral gives us:

$$
\begin{aligned}
&\cos(\omega(x-s)) - 2\cos(\omega x) + \cos(\omega(x+s))\\
=&\cos(\omega x - \omega s)) - 2\cos(\omega x) + \cos(\omega x + \omega s)\\
=&\cos(\omega x)\cos(\omega s) - \cancel{\sin(\omega x)\sin(\omega s)} - 2\cos(\omega x)\\
&+ \cos(\omega x)\cos(\omega s) + \cancel{\sin(\omega x)\sin(\omega s)}\\
=&2\cos(\omega x)(\cos(\omega s) - 1)
\end{aligned}
$$

Cosine initial condition

Continuing calculation:

$$
\begin{aligned}
&\Delta^{\alpha/2}u(t,x)\\
=&f(t)\int_0^\infty \frac{2\cos(\omega x)(\cos(\omega s)-1)}{s^{1+\alpha}}ds\\
=&2f(t)\cos(\omega x)\int_0^\infty \frac{\cos(\omega s)-1}{s^{1+\alpha}}ds\\
=&\text{(trust me)}\\
=&2f(t)\cos(\omega x)|\omega|^\alpha \cos\left(\frac{\alpha\pi}{2}\right)\Gamma(-\alpha)
\end{aligned}
$$

Without loss of generality we may assume $\omega \geq 0$:

$$
\Delta^{\alpha/2}u(t,x) = 2f(t)\cos(\omega x)\omega^\alpha \cos\left(\frac{\alpha\pi}{2}\right)\Gamma(-\alpha)
$$

Cosine initial condition

Substituting these results into fractional diffusion equation:

$$\frac{\partial u(t,x)}{\partial t} = a^2 \Delta^{\alpha/2} u(t,x)$$

$$\frac{df(t)}{dt}\cos(\omega t) = a^2 \cancel{c_{1,\alpha}} \cancel{2} f(t)\cos(\omega x)\omega^{\alpha} \cancel{\cos\left(\frac{\alpha\pi}{2}\right)\Gamma(-\alpha)}$$

$$\frac{df(t)}{dt}\cos(\omega t) = a^2 f(t)\cos(\omega x)\omega^{\alpha}$$

which leads to ordinary differential equation for f:

$$\frac{df(t)}{dt} = a^2 \omega^{\alpha} f(t).$$

Cosine initial condition

The equation for scaling function f can be solved explicitly:

$$\frac{df(t)}{dt} = -f(t) \cdot a^2 \omega^\alpha$$
$$f(t) = f(0) \exp\left(-a^2 \omega^\alpha t\right)$$

with $f(0) = A_\omega$, to match initial condition $u(0, x) = A_\omega \cos(\omega x)$.
Therefore a solution to diffusion equation is:

$$u(t, x) = A_\omega \exp\left(-a^2 \omega^\alpha t\right) \cos(\omega x)$$

Fourier series

The diffusion equation is *linear*. If solution for initial condition $u_0(x) = A_\omega \cos(\omega x)$ has form:

$$u(t,x) = A_\omega \exp\left(-a^2\omega^\alpha t\right)\cos(\omega x),$$

then solution for initial condition: $u_0(x) = \sum_i A_{\omega_i} \cos(\omega_i x)$ has form:

$$u(t,x) = \sum_i A_{\omega_i} \exp\left(-a^2\omega_i^\alpha t\right)\cos(\omega_i x).$$

Numerical results

We applied the method to initial condition

$$u_0(x) = \cos(x/2) + \cos(x) + \cos(10x).$$

You will notice "wrinkles" vanishing faster when α goes higher.
The method is applicable to any initial condition with Neumann boundary conditions:

$$\left.\frac{\partial u(t,x)}{\partial x}\right|_{x=0} = 0 = \left.\frac{\partial u(t,x)}{\partial x}\right|_{x=L},$$

with L being period of the function in x direction.

Numerical results

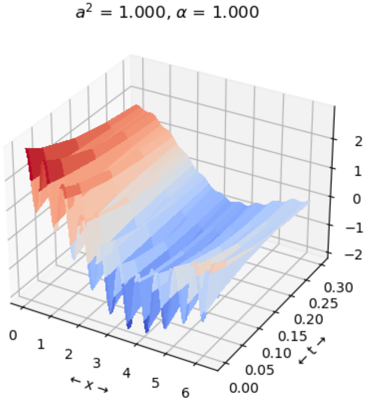


Fig. 5. Diffusion of order $\alpha = 1$

Numerical results

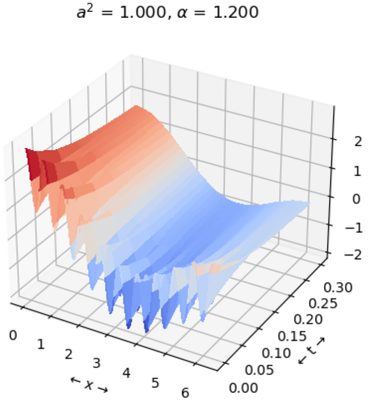


Fig. 6. Diffusion of order $\alpha = 1.2$

Numerical results

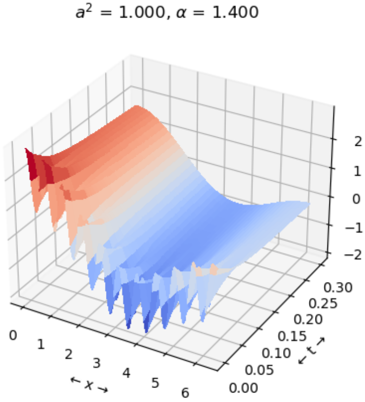


Fig. 7. Diffusion of order $\alpha = 1.4$

Numerical results

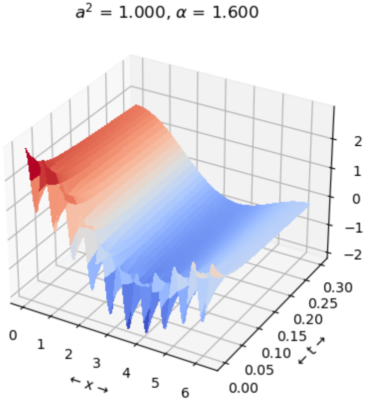


Fig. 8. Diffusion of order $\alpha = 1.6$

Numerical results

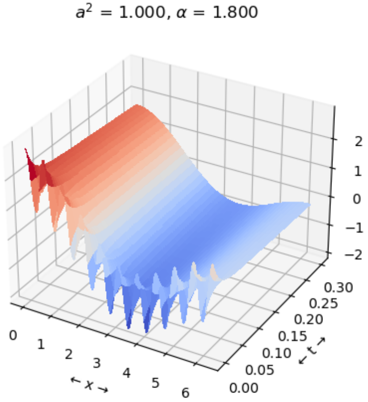


Fig. 9. Diffusion of order $\alpha = 1.8$

Numerical results

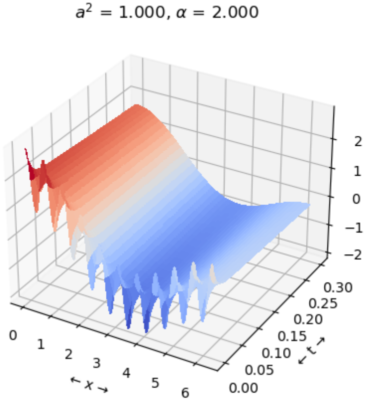


Fig. 10. Diffusion of order $\alpha = 2.0$

Numerical methods

For more general diffusion, we discredized x-axis into sub-intervals with length dx such that there are 256 sub-intervals. We used midpoint of each interval as representatives.

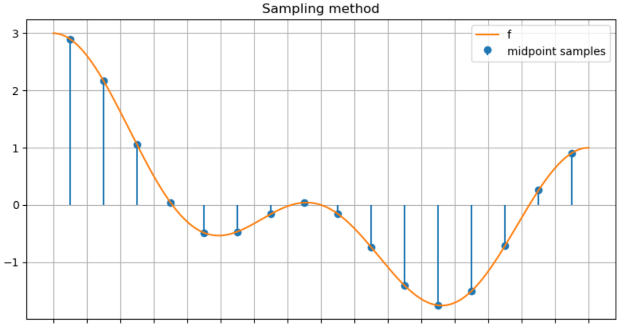


Fig. 11. Example of midpoint discretization on 16 sub-intervals

Numerical methods

Next, we constructed a system of 256 ordinary differential equations:

$$
\begin{aligned}
\frac{d}{dt}u(t, x = dx/2) &= a^2 \Delta^{\alpha/2} u(t,x)\big|_{x=dx/2} \\
\frac{d}{dt}u(t, x = 3dx/2) &= a^2 \Delta^{\alpha/2} u(t,x)\big|_{x=3dx/2} \\
\frac{d}{dt}u(t, x = 5dx/2) &= a^2 \Delta^{\alpha/2} u(t,x)\big|_{x=5dx/2} \\
&\vdots \\
\frac{d}{dt}u(t, x = 511dx/2) &= a^2 \Delta^{\alpha/2} u(t,x)\big|_{x=511dx/2}
\end{aligned}
$$

This system was solved by Runge-Kutta method. Other numerical method would also be suitable.

Numerical methods

To evaluate $\Delta^{\alpha/2}u(t,x)$ at all requiered points, we used discrete cosine transform type II. This is a efficient way to represent sampled function as a sum of cosines.
If the function is periodic with period L, then DCT can calculate coeffiecient A_ω for $\omega \in \left\{0, \frac{\pi x}{L}, \frac{2\pi x}{L}, \ldots\right\}$ such that $f(x) = \sum_\omega A_\omega \cos(\omega x)$ at each sampled point.
This way we could evaluate fractional Laplacian on right-hand side very fast. To evaluate it at all n points we need $\mathcal{O}(n \log n)$ operations, which is much better than $\mathcal{O}(n^2)$ operations with naïve approach.

Future work

Our aim is to apply this diffusion process to epidemiological model in which non-linear terms are present.
Susceptible individuals can become infectious after encountering already infectious individual. Infectious individuals overcome disease and get immunity spontaneously. This process can be described by transitions:

$$S \xrightarrow{\beta SI} I$$

$$I \xrightarrow{\gamma I} R$$

Model is formulated by system of differential equations:

$$\frac{\partial S(t,x)}{\partial t} = -\beta S(t,x) I(t,x) + \Delta^{\alpha/2} S(t,x)$$
$$\frac{\partial I(t,x)}{\partial t} = \beta S(t,x) I(t,x) - \gamma I(t,x) + \Delta^{\alpha/2} I(t,x)$$
$$\frac{\partial R(t,x)}{\partial t} = \gamma I(t,x) + \Delta^{\alpha/2} R(t,x)$$

Thank you for your attention