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NP-completeness and degree restricted spanning trees

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Abstract

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In this paper several results are proved: (1) Deciding whether a given planar graph G with maximum degree 3 has a spanning tree T, where $\deg(x, T) = 1$ or 3 for each node x, is NP-Complete. (2) For each proper subset S of the positive integers $\mathbb{Z}^+$ with $1 \in S$ and $|S| \geqslant 2$, deciding whether a planar graph $G = (V, E)$ has a spanning tree T such that $\deg(x, T) \in S$, for all $x \in V$, is NP-Complete. (3) For each non-empty proper subset S of $\mathbb{Z}^+$, deciding whether, given a planar graph $G = (V, E)$ and integer N with $1 \leqslant N < |V|$, there is a spanning tree T for G such that T has at least [at most] N nodes x with $\deg(x, T) \in S$ is NP-complete. Also, as corollaries of (1), we show that for a planar graph G, with n nodes and maximum degree 3, it is NP-complete to decide (4) if there is a spanning tree with at least [exactly] $n/2 + 1$ leaves and (5) if G has a connected dominating set with cardinality $\leqslant n/2 - 1$.

1. Introduction

In this paper we establish the following results.

Theorem 1. *Given any planar graph $G = (V, E)$ with maximum degree 3, it is NP-complete to decide if there exists a spanning tree T for G such that $\deg(x, T) = 1$ or 3, for all $x \in V$.*

Corollaries. *Let G be a planar graph G with n nodes and maximum degree 3.*

(1) *It is NP-complete to decide if there is a spanning tree for G with at least $n/2 + 1$ leaves. Furthermore, any such spanning tree must have exactly $n/2 + 1$ leaves (in fact it must be a $\{1, 3\}$-tree).*

(2) *It is NP-complete to decide if there is a connected dominating set D for G with cardinality no greater than $n/2 - 1$. Furthermore, any such D must have cardinality $= n/2 - 1$. (A connected dominating set D for a graph $G = (V, E)$ is a subset of V which induces a connected subgraph of G and where every node in $V - D$ is edge connected to some node in D.)*

Theorem 2. *For any fixed proper subset S of the positive integers $\mathbb{Z}^+$, with $1 \in S$ and $|S| \geq 2$, and for any planar graph $G = (V, E)$, it is* NP-*complete to decide if there exists a spanning tree T for G such that* $\deg(x, T) \in S$, *for all $x \in V$.*

Theorem 3. *For any non-empty proper subset S of $\mathbb{Z}^+$, and any pair (G, N), with $G = (V, E)$ a planar graph and N an integer with $1 \leq N < |V|$, it is* NP-*complete to decide if there exists a spanning tree T for G such that T has at least [at most] N nodes x with* $\deg(x, T) \in S$.

The proof of Theorem 1 involves a local replacement argument. Theorem 1 resolves an open question asked in [1]. Previously, it has been shown [2] that Theorem 2 holds when $S = [1, m]$ for any $m \in \mathbb{Z}^+$, and that Theorem 3 holds for $S = \{1\}$. Recently it has been shown [1] that Theorem 2 holds when $S = \mathbb{Z}^+ - \{2\}$. Theorems 2 and 3 extend NP-completeness to all other (allowed) sets S—finite or infinite.

It is known [2] that, given a constant K and a planar graph G with maximum degree 4, it is NP-complete to decide if there is a spanning tree for G with at least K leaves. Corollary 1 improves on this and answers the question posed by Joan Hutchinson [personal communication] "Is it NP-complete to decide if a graph with n nodes has a spanning tree with at least $n/2 + 1$ leaves?" Jerry Griggs and M. Wu have recently announced that every connected graph with minimum degree at least 5 has a spanning tree with at least $n/2 + 2$ leaves. And Kleitman and West have recently announced that as the minimum degree grows, so does the number of leaves in some spanning tree, approaching n leaves.

It is known [2] that, given a constant K and a planar graph G with maximum degree 4 it is NP-complete to decide if G has a connected dominating set of cardinality $\leq K$. Corollary 2 improves on this.

Notation and Definitions. (a) For any set V, $|V|$ denotes the cardinality of V.

(b) $\deg(x, T)$ is the degree of node x in the tree T.

(c) For any $m \in \mathbb{Z}^+$, $[1, m]$ denotes the closed interval of integers from 1 through m.

(d) We denote an edge $\{x, y\}$ in an undirected graph G by $[x, y]$.

(e) If $S \subseteq \mathbb{Z}^+$, an S-tree T is a tree in which $\deg(x, T) \in S$ for all nodes x of T.

(f) NPC denotes 'NP-complete'.

2. Proofs of the theorems and corollaries

In order to prove Theorem 1 we must first establish a lemma.

Lemma. *Consider the problem* HPV *('Hamiltonian Path Variant') where we are given a connected planar graph G, for which there are exactly two nodes A and B*

of degree 1 and all other nodes have degree 3, and we ask if there is a Hamiltonian path (between A and B) in G. HPV *is an* NPC *problem.*

Proof. This follows by slightly altering the proof in [3] that it is NPC to decide if there is a Hamiltonian circuit in a planar, cubic, 3-connected, undirected graph. Using the notation and terminology in [3], we only need alter the authors' construction as follows: Delete the two-input 'or' graph between the two edges $\{v_{11}, w_{11}\}$ and $\{v_{n4}, v_{m6}\}$, delete edge $\{v_{11}, w_{11}\}$, and add edges $\{w_{11}, A\}$ and $\{v_{11}, B\}$. With these changes, Fig. 7 in [3] becomes Fig. 3 (in the present paper). The original graph will have a Hamiltonian circuit if and only if the new graph has a Hamiltonian path (necessarily between A and B). Also the new graph is planar and all its nodes have degree 3 except for the degree 1 nodes A and B. □

Proof of Theorem 1. Let $P^{1,3}$ denote the problem where we are given a planar graph G with all nodes of degree ≤ 3 and ask if G has a spanning $\{1,3\}$-tree. Clearly $P^{1,3}$ is in NP. HPV is transformed to $P^{1,3}$. Let $G=(V, E)$ be any connected planar graph with two nodes x and y of degree 1 and all other nodes with degree 3. We locally replace each degree 3 node in G, using the transformation in Fig. 1, to obtain a graph $G'=(V', E')$ having maximum node degree 3. The graph in Fig. 1(b) that replaces the node in Fig. 1(a) will be referred to as the 'expansion of the node'. Obviously the construction can be accomplished in polynomial time. We claim that there exists a Hamiltonian path P between x and y in G if and only if there is a spanning $\{1,3\}$-tree T for G'.

$\Rightarrow$: If P uses edges 1 and 2 [respectively edges 1 and 3] in Fig. 1(a), then let T use the heavy edges in Fig. 2(a) [Fig. 2(b)]. The case where P uses edges 2 and 3 is symmetric to the case where edges 1 and 3 are used. We therefore obtain a spanning $\{1,3\}$-tree T for G' from the Hamiltonian path P in G.

$\Leftarrow$: Consider which edges T can use in Fig. 1(b). T must use edges 4, 5, and 12. T cannot use all the edges 1, 2, and 3 or else, due to the degree restriction and spanning condition, T would have to contain edges 4 through 9, a circuit, which is

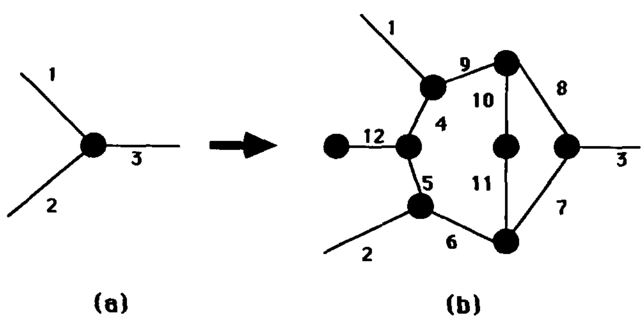

Fig. 1. Transformation replacing a degree 3 node with a graph having maximum degree 3.

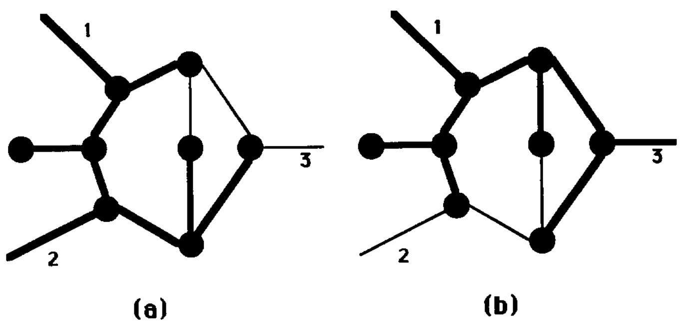

Fig. 2. The heavy edges in (a) [respectively (b)] show the edges in an 'expanded node' that belong to the $\{1,3\}$-tree T in G' for the case when the Hamiltonian path in G uses edges 1 and 2 [1 and 3] in Fig. 1a.

a contradiction. T cannot use only one of the edges $1,2,3$ or else T either does not span all nodes in Fig. 1(b) or else violates the degree restrictions. Therefore T must use exactly two of the edges $1,2,3$. Edges 10 and 11 act as a '*switch*'. T must contain one of these edges, but not both. If T uses edge 10, then T uses edge 1 and either 2 or 3; if T uses edge 11, then T uses 2 and either 1 or 3. Furthermore, it is critical to the proof [in order to allow a 'collapse back' of Fig.

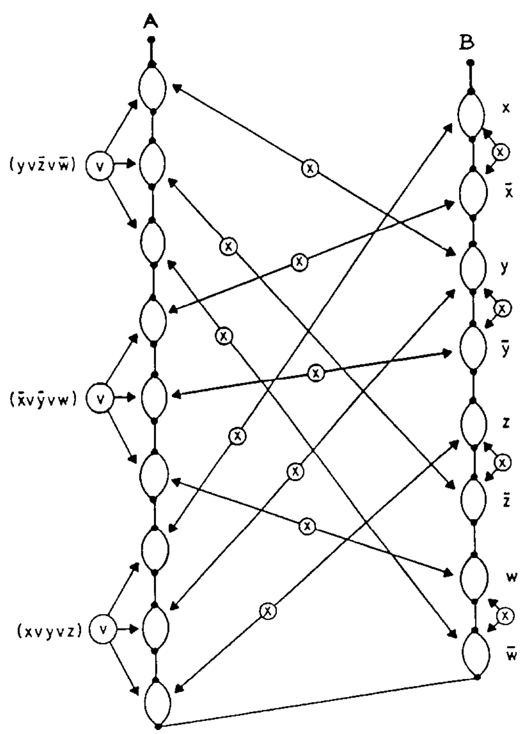

Fig. 3. Fig. 7 in [3] modified for proving the Lemma.

1(b) to Fig. 1(a)] to realize that in all these two edge possibilities the intersection of T-{edges $1, 2, 3$} with Fig. 1(b) results in a *connected* graph (of 8 nodes). Thus we can collapse Fig. 1(b) back to Fig. 1(a) resulting in T being converted to a connected spanning subgraph P of G in which every degree 3 node in G has exactly two of its edges in P and in which x and y have degree 1 in P. P must therefore be a desired Hamiltonian path in G. Thus Theorem 1 is proved. □

Proof of Corollary 1. Let G be a planar graph with n nodes and maximum degree 3. Also, define $K = n/2 + 1$ and let T be a spanning tree for G. Then the following are equivalent:

(i) T is a $\{1, 3\}$-tree,

(ii) T has at exactly K leaves,

(iii) T has at least K leaves.

Obviously (ii)⇒(iii). We also show that (i)⇒(ii) and (iii)⇒(i). Let n_i be the number of nodes x of T such that $\deg(x, T) = i$, for $i = 1, 2, 3$.

(i)⇒(ii): We have $n_1 + n_3 = n$ and $3 * n_3 + 1 * n_1 = 2 * (n-1)$. It follows that $n_1 = K$.

(iii)⇒(i): We have $n_1 + n_2 + n_3 = n$, $3 * n_3 + 2 * n_2 + 1 * n_1 = 2 * (n-1)$, and $n_1 \geq K$. It follows that $n_2 = 0$, i.e., T is a $\{1, 3\}$-tree.

Corollary 1 is now immediate from Theorem 1. □

Proof of Corollary 2. Let G be a planar graph with n nodes and maximum degree 3. Also, define $K = n/2 - 1$. We claim that there is a spanning tree T for G with at least $n - K$ leaves if and only if there is a connected dominating set D for G with $|D| \leq K$. Also, any such D must have $|D| = K$.

⇒: Just let D be the nonleaves of T.

⇐: Let T' be any spanning tree for the subgraph induced by D. It is easy to see that the nodes of G not in D may be added as leaves to T' to give a spanning tree T for G.

That any such D must have $|D| = K$ follows from Corollary 1 as $n - K = n/2 + 1$. Therefore our claim is true. Corollary 2 now follows from Corollary 1. □

Proof of Theorem 2. Let the problem posed in Theorem 2 be denoted by P_2. P_2 is clearly in NP.

Case 1: $S = [1, m]$ *for some* $m \geq 2$.

This is known [2] to be NPC. Indeed, for this case we may restrict the graph to have maximum degreee $m + 1$: We transform the known [3] NPC problem 'Hamiltonian path for cubic planar graphs' to an instance of P_2, case 1, with maximum degree $m + 1$. Let G be any cubic planar graph. Attach $m - 2$ edges (i.e., degree 1 nodes) to each node of G to obtain a graph G'. Thus each node of G' has degree $m + 1$. G has a Hamiltonian path if and only if G' has a spanning S-tree: The 'only if direction' is obvious; the 'if direction' follows as any spanning

S-tree $T'$of G' yields (by deleting the new edges) a spanning tree T in G where all the nodes of T have degree 1 and 2, whence T must be a Hamiltonian path in G.

Case 2: *S is not an interval,* i.e., *$1 \in S$ and there exists an $m \geq 2$ such that $m \notin S$ and $m+1 \in S$.*

We transform an instance G of HPV to an instance of P_2, case 2 (which in fact will have maximum degree $m+2$ where m may be taken as the smallest integer satisfying Case 2). Add $m-1$ edges to each node in G, except to the two degree 1 nodes x and y in G to obtain a graph G'. There is a Hamiltonian path between x and y in G if and only if there is a spanning S-tree T' in G': If H is a Hamiltonian path between x and y in G, then H together with the added edges is clearly a spanning S-tree in G'. Conversely, if T' is a spanning S-tree in G', let H be T' with the added edges removed. H is a spanning tree for G. x and y are the only degree 1 nodes in H as T' is an S-tree and $m \notin S$. But the only tree with exactly two leaves is a path, i.e., H is a Hamiltonian path in G between x and y. □

The above proof of Case 2, due to Joan P. Hutchinson [personal communication], is simpler than the author's original proof of this case.

Proof of Theorem 3. Call the version of the problem posed in Theorem 3 where we use 'at least N nodes' [respectively, 'at most N nodes'] version A [version B]. Version B is NPC if and only if Version A is NPC as an instance $(G=(V, E), N)$ of Version B with respect to set S has a yes answer if and only if instance $(G, |V|-N)$ of Version A has a yes answer with respect to set $\mathbb{Z}^+ - S$. We now show that Version A is NPC. Let $P[S]$ denote Version A with respect to set S.

Case 1: $1 \notin S$.

Let $G=(V, E)$ be a planar graph and let $N=|V|$-2. Then case 1 is equivalent to asking if there is a spanning tree T for G with $\leq$2 nodes not in S, which is equivalent to saying T has exactly 2 leaves, i.e., T is a Hamiltonian path in G. This concludes case 1.

Case 2: *$1 \in S$ and there exists an integer $m \geq 2$ such that $m \notin S$ and $m+1 \in S$.*

Let $S' = \{i-(m-1)\colon i \in S, i > m\}$. Thus $1 \notin S'$ and $S' \neq \emptyset$. $P[S']$ is NPC by case 1. We transform $P[S']$ to $P[S]$. Let $(G'=(V', E'), K')$ be an arbitrary instance of $P[S']$. Attach $m-1$ edges to each node in G' to obtain a planar graph $G=(V, E)$. Define $K = K' + |V'| * (m-1)$. There is a spanning tree for G' with $\geq K'$ nodes having degree in S' if and only if there is a spanning tree for G with $\geq K$ nodes having degree in S. (Note that every spanning tree for G has exactly $|V'| * (m-1)$ leaves and has no nonleaf nodes of degree $<m$.) This concludes case 2.

Case 3: *$S=[1, \ldots, m]$ for some $m \geq 1$.*

The $m=1$ case, as was previously mentioned, is known to be NPC. (Of course Corollary 1 is a stronger statement). Let $m \geq 2$. We show that $P[\{1\}]$ transforms to $P[S]$. Let $(G' = (V', E'), K')$ be an instance of $P[\{1\}]$. Attach $m - 1 \geqslant 1$ edges to each node of G' to obtain a planar graph G. Define $K = K' + |V'| * (m - 1)$. There is a spanning tree for G' with $\geqslant K'$ leaves if and only there is a spanning tree for G with $\geqslant K$ nodes having degree in S. (Note that any spanning tree T for G has exactly $|V'| * (m - 1)$ leaves and (if $K' > 1$) any nonleaf node of T with degree in S must have degree m.) This concludes case 3. □

Note. The author has learned that Lemke [4] has proved (using a different argument) Theorem 1 and Corollary 1 for cubic graphs.

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