Spanning trees in graphs

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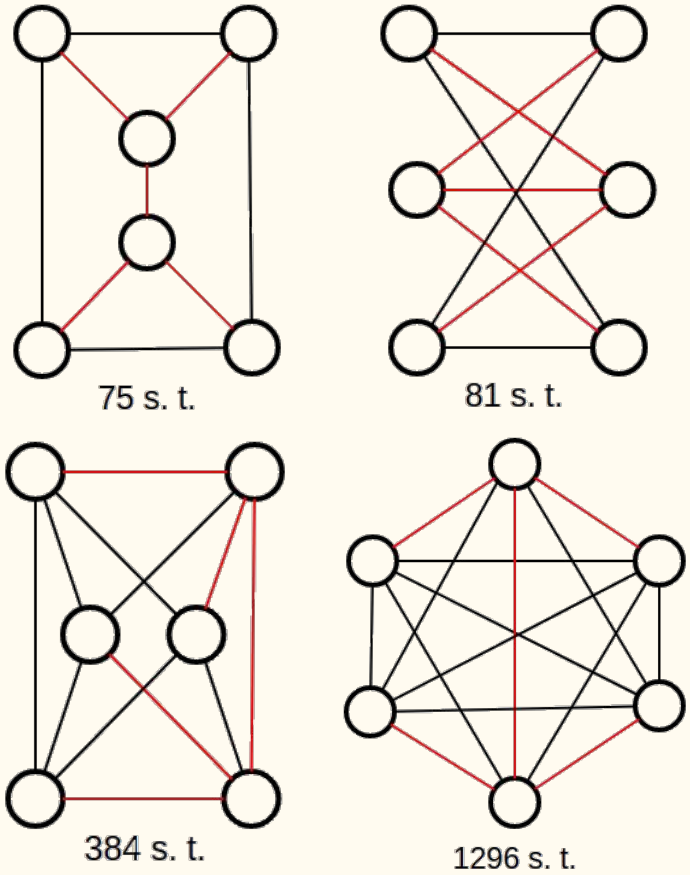
Graphs and spanning trees

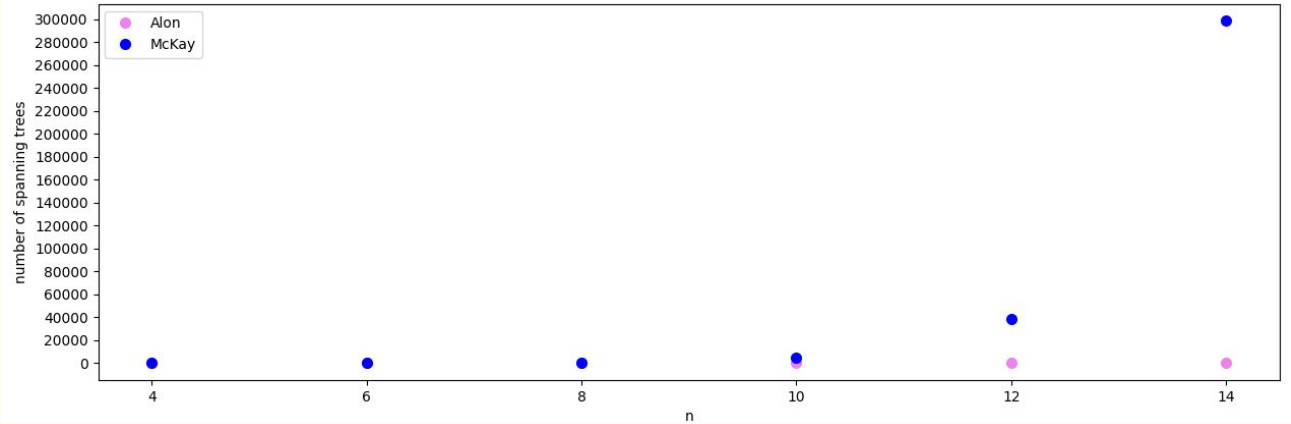
- undirected simple connected graphs
- spanning tree of connected graph G
  - maximal set of edges of G with no cycles
- focus on k-regular graphs
  - all vertices have k neighbours



regular graphs and their spanning trees

Motivation

- easy to count spanning trees in a particular graph, but not in a whole class of graphs → estimations needed
- k-regular graphs on n vertices
  - Noga Alon: The Number of Spanning Trees in Regular Graphs
  - Brendan McKay: Spanning Trees in Regular Graphs

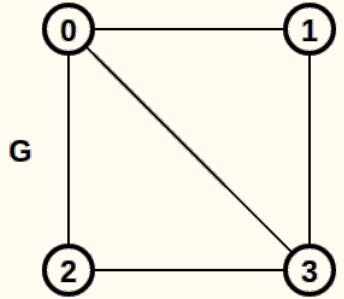


Experiments

- identify graphs with minimum and maximum number of spanning trees in a specified set of graphs
  - k-regular graphs on n vertices (k= 3; 4)
  - graphs on n + 1 vertices, n vertices are of degree k_1, one vertex is of degree k_2, $k_1 \neq k_2$
- compare the numbers of labeled and unlabeled spanning trees in a pair of graphs

Implementation

- graph generation - genreg
- graph processing - C++
- running experiments - bash scripts
  - combines graph generation and processing
- spanning tree counting - Kirchhoff’s Theorem



$$Q = \begin{Bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{Bmatrix}$$

Methods for graph generating and processing

challenges: time and memory

<typeOfGeneration>serial n k []

- runs generation and processing of k-regular graphs on n vertices

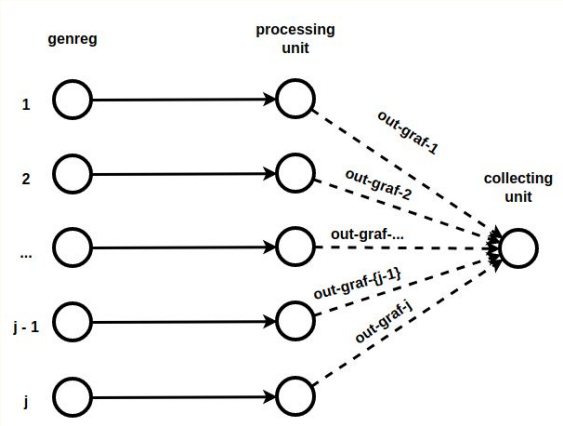
```
min 1 75
[(0, 1), (0, 2), (0, 3), (1, 2), (1, 4), (2, 5), (3, 4), (3, 5), (4, 5)]
max 1 81
[(0, 1), (0, 2), (0, 3), (1, 4), (1, 5), (2, 4), (2, 5), (3, 4), (3, 5)]

          GENREG - Generator fuer regulaere Graphen
          6 Knoten, Grad 3, Taillenweite mind. 3
          Erzeugung gestartet...
          2 Graphen erzeugt.
          Laufzeit:0.0s
Generating and processing finished after 0 seconds
```

```
terezia@terezia-ntb:~/genreg/grafy/ukazka$ regularSerial 20 3
3-regular graphs on 20 vertices to file maxMinReg3-20.txt
processing finished after 18 seconds
Generating and processing finished after 18 seconds
```

<typeOfGeneration>paralel j n k []

- splits generation and processing into j parts



Overview of results

- processed sets of graphs graphs
  - 3-regular graphs - on up to 28 vertices (40 497 138 011 graphs)
  - 4-regular graphs - on up to 19 vertices (11 946 487 647 graphs)
- hypothesis about 3-regular graphs with minimum and maximum number of spanning trees
- estimation for maximum number of spanning trees for 3-regular graphs on up to 42 vertices

generation+processing times of sets of 3-regular graphs on n vertices

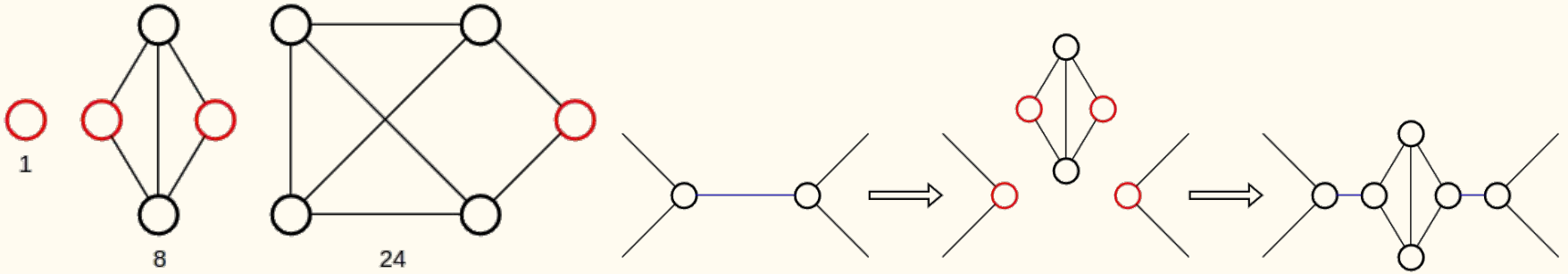
n	time
16	0,075s
18	0,398s
20	4,711s
22	1min 11,581s
24	21min 24,074s
26	2.861 hr
28	$\sim$ 4 days

3-regular graphs with minimum number of spanning trees

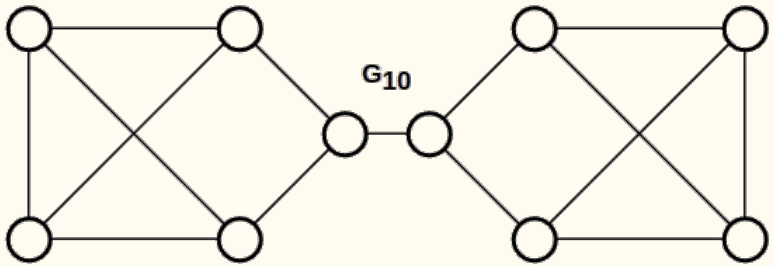
- formed from building blocks determining the number of spanning trees
- hypothesis for the number of spanning trees based on iterative construction:

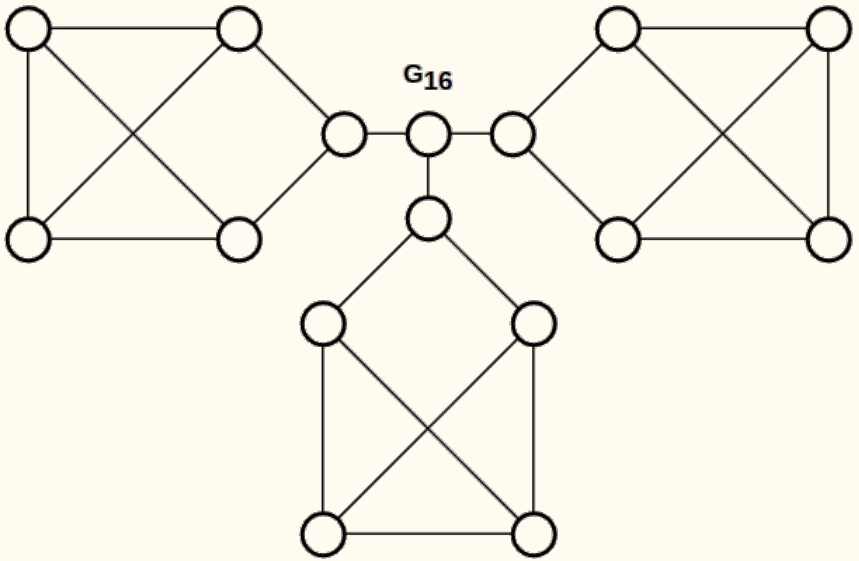
  $24^2 \cdot (8^{(n-2\cdot5)/4})$ spanning trees for $n = 10 + 4i$, $i \in N$

  $24^3 \cdot (8^{(n-3\cdot5-1)/4})$ spanning trees for $n = 16 + 4i$, $i \in N$



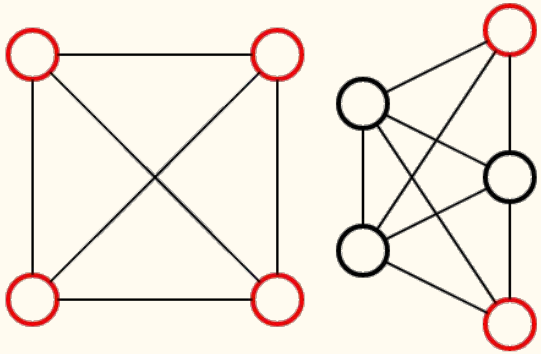
3-regular graphs with minimum number of spanning trees

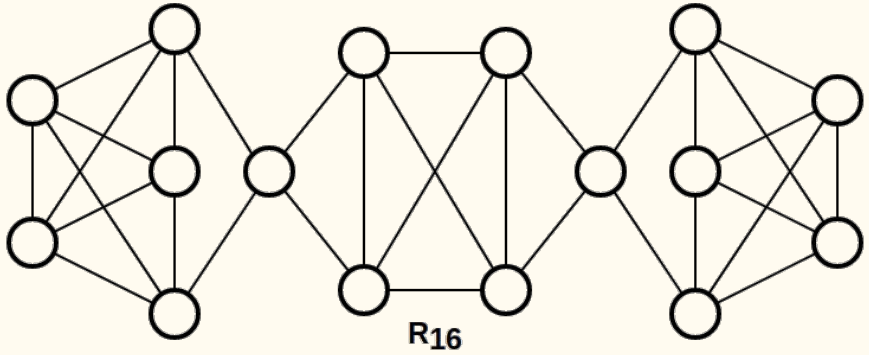




4-regular graphs with minimum number of spanning trees

- pair of building blocks + additional vertices
- no bridges present → more complex structure of graphs and spanning tree counting



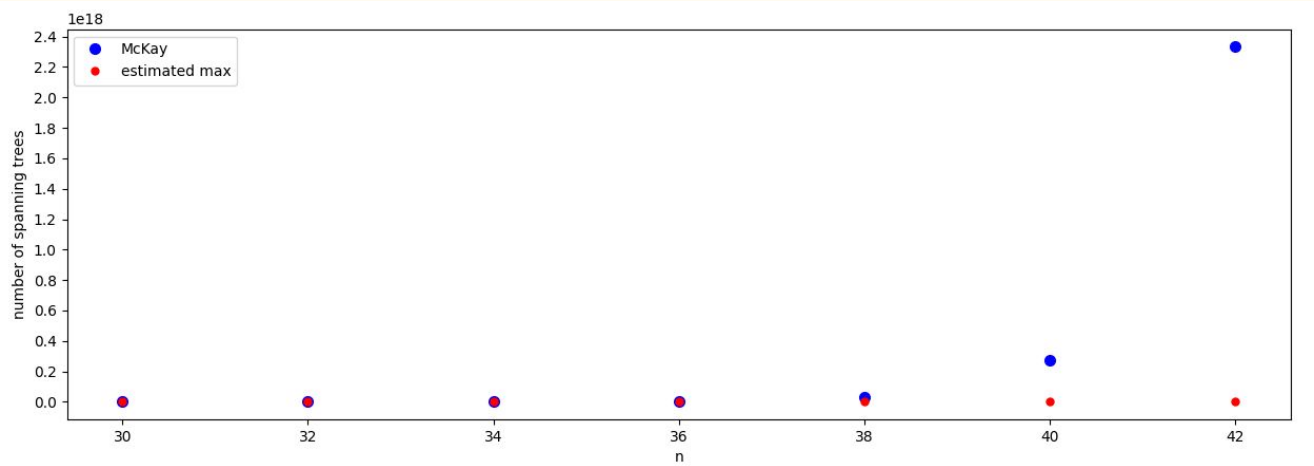


3-regular graphs with maximum number of spanning trees

- highest possible girth for the given n (girth - length of the shortest cycle in the graph)

  ↳ enables for estimations for higher values of n

- for n = 4, 6, 10, 14, 24 and 30, the graphs are cages
  - regular graphs with the least possible number of vertices for a given girth



Main contributions

- personal - work with large sets of graphs
- if our hypothesis for the minimum number of spanning trees in 3-regular graphs is correct, it is more accurate than Alon’s lower bound
- relation between graphs with minimum/maximum number of spanning trees and other areas from graph theory (girth, cages…)

Thank you for your attention

Bibliography

MCKAY B., Spanning Trees in Regular Graphs. Europ. J. Combinatorics (1983) 4. 1983. 149-160.

ALON N., The Number of Spanning Trees in Regular Graphs. Random Struct. Algorithms 1(2). 1990. 175-182.